Quanto Adjustments in the Presence of Stochastic Volatility

Alexander Giese
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Quanto Adjustments in the Presence of Stochastic Volatility

Alexander Giese
UniCredit
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Abstract

This paper considers the pricing of quanto options in the presence of stochastic volatility. While it is well known that the quanto adjustment in the drift of the underlying has a significant impact on the prices of quanto options, this paper points out that an additional quanto adjustment in the underlying's volatility needs to be considered in the presence of stochastic volatility. By deriving closed-form solutions for standard quanto options, the paper demonstrates that this additional quanto adjustment also has a material impact on quanto options. Furthermore, numerical examples are presented together with a comparison of the proposed model against three commonly used standard pricing methods for quanto options.
1. Introduction

Quanto options are options where the payoff is paid in a currency different from the currency in which the underlying asset is traded and where the applied foreign exchange (FX) rate between the two currencies is set to one. The fixed forex rate allows the holder of a quanto option to participate in the performance of the underlying without carrying the risk of a changing forex rate. For instance for a Euro-based investor who is seeking option exposure on the S&P 500 but does not want to be exposed to changes of the Euro/US Dollar exchange rate, a quanto option on the S&P 500 is a very suitable financial product as it pays the payoff of a standard non-quanto option on the S&P 500 and converts the payout with a guaranteed rate of 1 from US Dollar into Euro at maturity. Quanto options are traded as over-the-counter (OTC) contracts and are also often embedded in structured equity products offered to end investors due to the increasing globalization of equity investments.

Pricing and risk-managing quanto options on foreign equities has become increasingly challenging in recent years due to unpredicted levels of the equity/forex correlations and high volatilities. Both market parameters determine the well-known quanto adjustment in the drift of the underlying, as derived by Reiner [8] in the classical Black-Scholes model. While most of the research on quanto options has focused on the Black-Scholes framework, researchers recently started to study quanto options in the context of stochastic volatility models, which allow to incorporate skews and smiles in the implied volatility surface of the underlying asset. Dimitroff et al. [1] assume the Heston [3] model and Jäckel [5] uses a stochastic local volatility model in their studies on quanto options. While both studies conclude that the quanto option prices in a stochastic volatility model differ from the corresponding prices obtained by applying standard pricing methods, they provide little explanation or intuition for the observed price differences. Furthermore, in both papers the model prices for quanto options need to be calculated using either Monte Carlo methods or numerical solutions of the pricing partial differential equation (PDE) due to the absence of closed-form solutions.

Motivated by these recent numerical studies of quanto options in the presence of stochastic volatility, we aim to obtain closed-form solutions for standard quanto options under the assumption of a stochastic volatility model for the underlying asset in order to facilitate fast and efficient pricing and risk management of these options. We also try to provide a good understanding and intuition for the main factor causing the price differences between the quanto option prices obtained using the derived pricing formulas and the option prices obtained by using standard pricing methods for quanto options.

The remainder of this paper is organized as follows. We first introduce the stochastic volatility model and derive closed-form solutions for the quanto forward in the model framework. Closed-form solutions for standard quanto options are derived in Section 3 which represents the main result of the paper. Afterwards, Section 4 discusses the calibration of the model and analyzes the impact of an additional quanto adjustment which we identify to be present. Section 5 presents numerical examples where the model prices are compared against three commonly used pricing methods for quanto options.
Furthermore, a numerical example for the impact of the implied volatility skew of foreign exchange options on the prices of quanto options is given. Finally, Section 6 concludes the paper.
2. The Model

The price process of the underlying $S$ is assumed to be denominated in the foreign currency $X$ and to follow the dynamics:

$$ dS(t) = \left(r^X - d\right)S(t)dt + \nu(t)S(t)\,dW^X_S(t), \quad S(0) = S_0, $$

$$ dv(t) = \kappa(\theta - v(t))\,dt + \delta\,dW^X_v(t), \quad v(0) = v_0, $$

is under the foreign risk-neutral measure $Q^X$ where $W^X_S$ and $W^X_v$ are two Brownian motions, $r^X$ is the foreign interest rate, $d$ is the dividend yield and $\nu$ is the stochastic volatility process of the underlying $S$ with the constant parameters $\kappa$ (mean reversion speed), $\theta$ (long-term mean volatility) and $\delta$ (volatility of volatility). Here we assume the stochastic volatility model of Schöbel and Zhu [9] for the underlying price process where the volatility $v$ follows an Ornstein-Uhlenbeck process. This model choice will allow us later to derive closed-form solutions for standard quanto options, however, we strongly believe that most of the observations and conclusions of this paper apply to stochastic volatility models in general.\(^1\)

Furthermore, we assume an investor whose domestic currency is $Y$ and who wishes to obtain exposure to the underlying $S$ without carrying forex risk. Let $Z^{Y/X}$ denote the foreign exchange rate (price of one unit of currency $Y$ in units of currency $X$) and we assume $Z^{Y/X}$ is given by Black-Scholes model dynamics under $Q^X$:

$$ dZ^{Y/X}(t) = \left(r^Y - r^X\right)Z^{Y/X}(t)\,dt + \sigma_{fx}Z^{Y/X}(t)\,dW^X_Z(t), \quad Z^{Y/X}(0) = Z_0^{Y/X}, $$

where $W^X_Z$ is a Brownian motion, $r^Y$ is the domestic interest rate and $\sigma_{fx}$ is the constant volatility of the forex rate process $Z^{Y/X}$. The model allows for constant correlations between all driving factors, i.e.\(^2\)

$$ d\left[W^X_S, W^X_v\right](t) = \rho_{S,v}dt, \quad d\left[W^X_S, W^X_Z\right](t) = \rho_{S,Z}dt, \quad d\left[W^X_v, W^X_Z\right](t) = \rho_{v,Z}dt. $$

After a change of measure from $Q^X$ to the the domestic risk-neutral measure $Q^Y$ with

---

\(^1\) The Schöbel and Zhu [9] model has often been criticized for allowing the instantaneous volatility $v$ to becoming negative. However, this does not pose any mathematical or numerical problem as the non-negativity constraint only needs to be imposed on the variance rather than the instantaneous volatility itself. For instance Lipton and Sepp [6] advocate using the Schöbel and Zhu [9] model rather than the popular Heston [3] model in most applications.

\(^2\) For brevity here we assume constant parameters. However, the model and the main results of this paper can be generalized to time-dependent stochastic volatility parameters and correlations.
Girsanov’s theorem implies that the processes $W_S^Y$, $W_v^Y$ and $W_{FX}^Y$ defined by

$$
\begin{align*}
\frac{dQ^Y}{dQ^X} &= \frac{Z^{Y/X}(t)}{Z^{Y/X}(0)} e^{(r^X-r^Y)t} = e^{-\frac{1}{2} \sigma^2_{FX} t + \sigma_{FX} W_X^Y(t)}, \\
\end{align*}
$$

are Brownian motions under the domestic measure $Q^X$. The measure $Q^Y$ is also often referred to as the quanto measure. One obtains the following dynamics of the processes $S$ and $v$ under $Q^Y$:

$$
\begin{align*}
    dS(t) &= \left( r^X - d - \rho_{S,FX} \sigma_{FX} \right) S(t) dt + \nu(t) S(t) dW_S^Y(t), \\
    dv(t) &= \left[ \kappa ( \theta - v(t) ) - \rho_{v,FX} \sigma_{FX} \delta \right] dt + \delta dW_v^Y(t), \\
    dZ^{X/Y}(t) &= \left( r^Y - r^X \right) Z^{X/Y}(t) dt + \sigma_{FX} Z^{X/Y}(t) dW_{FX}^Y(t),
\end{align*}
$$

with $\hat{\delta} = \theta - \rho_{v,FX} \sigma_{FX} \delta / \kappa$, $\rho_{S,FX} = -\rho_{S,Z}$, $\rho_{v,FX} = -\rho_{v,Z}$ and the forex rate $Z^{XY}$ denoting the price of one unit of currency $X$ in units of domestic currency $Y$ ($Z^{XY}(t) = 1 / Z^{YX}(t)$). Furthermore, the correlation matrix between $W_S^Y$, $W_v^Y$, $W_{FX}^Y$ is given by

$$
\begin{pmatrix}
    1 & \rho_{S,v} & \rho_{S,FX} \\
    \rho_{S,v} & 1 & \rho_{v,FX} \\
    \rho_{S,FX} & \rho_{v,FX} & 1
\end{pmatrix}.
$$

The equation (3) features the well-known change in the drift of the underlying $S$ under the quanto measure $Q^Y$ and the quanto adjustment drift term is determined by the equity/forex correlation, the forex volatility and the equity volatility. However, we observe in (4) that the drift of the stochastic volatility also changes under the quanto measure $Q^Y$ and that this additional quanto drift term depends on the correlation $\rho_{v,FX}$, the forex volatility and the volatility of volatility. Effectively, the long-term mean
volatility changes from $\theta$ to $\hat{\theta}$ which is expected to have a significant impact on the prices of quanto options.\(^3\)

Before we investigate the pricing of quanto options in the next section, we first seek to find the price of the quanto forward in the model posed above. The quanto forward $F^q(t,T)$ is a contract which pays the price of the foreign underlying $S$ at time $T$ converted with a fixed forex rate of one into to the currency $Y$. Thus, the quanto forward is given as the expected value of $S(T)$ under the measure $Q^Y$:

$$F^q(t,T) = E^{Q^Y}[S(T)]$$

$$= S(t)e^{(r-d)T-t} \times E^{Q^Y}\left[ e^{-\rho_{S,v}\sigma_{S,v}\int_0^T v(s)ds - \frac{1}{2}\sigma_{S,v}^2 \int_0^T v(s)^2 ds + \rho_{S,v} \int_0^T v(s)dW^Y_v(s) + \frac{1}{2} \rho_{S,v}^2 \int_0^T v(s)dv^Y(s)} \right]$$

$$= S(t)e^{(r-d)T-t} \times E^{Q^Y}\left[ e^{-\rho_{S,v}\sigma_{S,v}\int_0^T v(s)ds - \frac{1}{2}\sigma_{S,v}^2 \int_0^T v(s)^2 ds + \rho_{S,v} \int_0^T v(s)dW^Y_v(s) + \frac{1}{2} \rho_{S,v}^2 \int_0^T v(s)dv^Y(s)} \right]$$

where we expressed the Brownian motion $W^Y_S$ as

$$W^Y_S(t) = \rho_{S,v} W^Y_v(t) + \sqrt{1-\rho_{S,v}^2} W(t)$$

with $W$ being a $Q^Y$ - Brownian motion independent of $W^Y_v$ and used the tower property. According to (4) and Ito's Lemma we have

$$d\nu(t)^2 = 2\kappa\left(\frac{\delta^2}{2\kappa} + \hat{\theta}\nu(t) - \nu(t)^2\right)dt + 2\delta \nu(t)dW^Y_v(t)$$

and

$$\int_t^T v(s)dW^Y_v(s) = \frac{1}{2\delta}\left( v(T)^2 - v(t)^2 - \delta^2 (T-t) - 2\kappa \hat{\theta} \int_t^T v(s)ds + 2\kappa \int_t^T v(s)^2 ds \right). \quad (5)$$

Using the last equation we obtain for the quanto forward:

$$F^q(t,T) = S(t)e^{(r-d)(T-t) - \frac{\rho_{S,v}}{2\delta}(v(t)^2 + \sigma^2(T-t))} \times E^{Q^Y}\left[ e^{-\rho_{S,v}\sigma_{S,v}\int_0^T v(s)ds - \frac{1}{2}\sigma_{S,v}^2 \int_0^T v(s)^2 ds + \rho_{S,v} \int_0^T v(s)dW^Y_v(s) + \frac{1}{2} \rho_{S,v}^2 \int_0^T v(s)dv^Y(s)} \right]$$

and applying Lemma 1 of the appendix finally yields

\(^3\) In case the volatility of volatility is zero and the volatility process therefore deterministic, the quanto drift in the volatility process disappears and the equations above reduce to the well-known equations for the Black-Scholes model with time-dependent volatility.
\[ F_q(t, T) = S(t)e^{\left(\delta - r\right)(T-t) - \frac{\rho_{S,r} \delta \sigma_F}{\sigma_S} \left(\nu(t) - \nu(T)\right)} D(t, T, \nu(t), s_1, s_2, s_3) \]  \hspace{1cm} (6)

with

\[
s_1 = -\frac{1}{2} \left( \frac{2\kappa \rho_{S,r}}{\delta} - \rho_{S,r}^2 \right), \quad s_2 = \frac{\kappa \hat{\rho}_{S,r} + \rho_{S,F} \sigma_F}{\delta}, \quad s_3 = \frac{\rho_{S,r}}{2\delta}.
\]

The function \( D \) is given in Lemma 1. Since quanto forwards are often liquidly traded, the closed-form solution (6) allows us to calibrate the model quickly to market quotes for quanto forwards.
3. Quanto options

The purpose of this section is to derive closed-form solutions for standard quanto options within the model framework described in the previous section. Let $C^q(t,T,K)$ denote the price of a quanto call option with strike $K$ and maturity $T$. Then we have

$$C^q(t,T,K) = e^{-r^q(t-T)}E^{Q^Y}[(S(T) - K)^+]$$

$$= e^{-r^q(t-T)}F^q(t,T)Q^Y[S(T) > K]e^{-r^q(T-t)}KQ^Y[S(T) > K]$$

with $Q_1^Y$ defined by the Radon-Nikodym derivative

$$\frac{dQ_1^Y}{dQ^Y} = \frac{S(T)}{F^q(t,T)}.$$  

Thus, the quanto call option price can be written as

$$C^q(t,T,K) = e^{-r^q(t-T)}F^q(t,T)P_1e^{-r^q(T-t)}KP_2$$  \hspace{1cm} (7)$$

with suitable probabilities $P_1$ and $P_2$. In remainder of this section we aim to obtain closed-form solutions for $P_1$ and $P_2$. For this, we consider the corresponding characteristic functions $f_1$ and $f_2$:

$$f_1(\phi) = E^{Q^Y}[e^{-i\phi\ln S(T)}] \quad f_2(\phi) = E^{Q^Y}[e^{-i\phi\ln S(T)}]$$

Defining $x(t)=\ln S(t)$, we start with working on $f_1$:

$$f_1(\phi) = \frac{1}{F^q(t,T)}E^{Q^Y}[e^{(-1+i\phi)x(T)}]$$

Applying Ito's Lemma we obtain from (3):

$$dx(t) = \left(r^x - d - \rho_{S,F^x}\sigma_{F^x}v(t) - \frac{1}{2}v(t)^2\right)dt + \rho_{S,v}v(t)dW^v(t) + \sqrt{1-\rho_{S,v}^2}v(t)dW(t).$$

Using the independence of $W$ together with the tower property as well as equation (5) and Lemma 1 of the appendix we come to the result:
Analogously, we get for $f_2$:

$$f_2(\phi) = e^{i\phi [v_{t}\xi_{T-t} + \xi_{T-t}]} \times D(t, T, \nu(t), \tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$$

with

$$\tilde{s}_1 = \frac{\phi^2}{2}(1 - \rho_{S,v}^2) + \frac{i\phi}{2} \left(1 - \frac{2\kappa P_{S,v}}{\delta} \right), \tilde{s}_2 = \frac{i\phi}{2} \left( \frac{\kappa \hat{\theta} P_{S,v}}{\delta} + \rho_{S,FX} \sigma_{FX} \right), \tilde{s}_3 = \frac{i\phi}{2} \frac{P_{S,v}}{\delta}.$$

Having closed-form solutions for the characteristic functions $f_1$ and $f_2$ enables us to compute the probabilities $P_1$ and $P_2$ via Fourier inversion:

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left[ \frac{e^{-i\phi \ln K} f_j}{i\phi} \right] d\phi, \quad j = 1, 2. \quad (10)$$

In summary, the quanto call price equation (7) together with the explicit formulas (8), (9) for the characteristic functions $f_1$ and $f_2$ and equation (10) give a closed-form solution for standard quanto call options. The value of a European quanto put option $P^q(t, T, K)$ can be obtained using the put-call parity for quanto options:

$$P^q(t, T, K) = C^q(t, T, K) + e^{-r(T-t)} K - e^{-r(T-t)} F^q(t, T).$$

To the best of our knowledge this is the first paper to give closed-form formulas for standard quanto options in a stochastic volatility model framework which enables a fast and efficient pricing of these options also in the presence of stochastic volatility and avoids the deployment of Monte Carlo methods or numerical solutions of PDEs.

\footnote{We refer the reader to Lord and Kahl [7] for the numerical aspects of the Fourier inversion.}
4. Model calibration and the impact of additional quanto adjustment

In order to use the model for the pricing of quanto options the model needs to be calibrated to the liquidly traded benchmark instruments. These benchmark instruments are non-quanto standard options on the underlying $S$, standard options on the exchange rate as well as quanto forwards which are often traded for the major underlyings. The first step in the calibration of the model is the calibration of the stochastic volatility process defined in (2) to standard options on $S$ where the payoff is paid in the currency of the underlying. For this the closed-form solution for standard options derived in Schöbel and Zhu [9] can be used together with standard calibration techniques as described in Gerlich et al. [2] for instance. This step determines the parameters $v_0$, $\kappa$, $\theta$, $\delta$ and $\rho_{S,v}$. Furthermore, the forex volatility parameter $\sigma_{FX}$ is chosen to match the at-the-money implied volatility on the forex rate corresponding to the maturity of the quanto option. In the last step we calibrate the model to the quanto forward given in the market and corresponding to the maturity of the quanto option. We still have the two model parameters $\rho_{S,FX}$ and $\rho_{v,FX}$ for matching the given quanto forward. In order to simplify the parameter choices we set the correlation $\rho_{v,FX}$ to

$$\rho_{v,FX} = \rho_{S,v} \rho_{S,FX}$$  \hspace{1cm} (11)

which corresponds to the parsimonious parametric form of the correlation matrix used by Dimitroff et al. [1] and Jäckel [5]. Furthermore, the parametric form (11) is well supported by time series data. In Figure 1 the historical correlation between the VIX index and the US Dollar/Euro rate is plotted against the product of the historical correlation between the S&P 500 index and the VIX index and the historical correlation between the S&P 500 index and the US Dollar/Euro rate. The correlations for a specific day are calculated based on the returns of the last 100 trading days. As visible in Figure 1 the realized correlation between the FX rate and the equity volatility is consistently positive since November 2008 which has a volatility reducing effect under the quanto measure (see (4)). Furthermore, the realized correlation is most of the time very close to the correlation estimated using equation (11). Alternatively, one could also estimate the correlation $\rho_{v,FX}$ directly based on historical data.

After estimating the parameter $\rho_{v,FX}$, it remains to find the correlation $\rho_{S,FX}$ by applying equation (6) and a root-finding algorithm to match the quanto forward given by the market. An application of the suggested calibration procedure to the S&P 500 index and the US Dollar/Euro rate for a maturity of $T=3$ years and market data from May 27, 2011 yields the model parameters listed in Table 1.

It is worth noting that the choice of the correlation parameter $\rho_{v,FX}$ has a significant impact on the model prices of quanto options as it determines the magnitude of the quanto adjustment in the
volatility. Table 2 lists the price\(^5\) of a Euro quanto call option on the S&P 500 index with maturity of 3 years and strike equal to the spot of the S&P 500 index for different values of the correlation parameter \(\rho_{v,FX}\). For each choice of the parameter \(\rho_{v,FX}\) the correlation \(\rho_{S,FX}\) is chosen such that the quanto forward is fitted in order to facilitate a proper comparison. The second row of Table 1 corresponds to the parametric form (11), however, other values for the correlation \(\rho_{v,FX}\) yield very different prices for the quanto call option although the price of the quanto forward remains the same. We observe that the higher the correlation \(\rho_{v,FX}\), the lower the quanto call option prices which can be well explained by the fact that a higher correlation \(\rho_{v,FX}\) results in a lower long-term mean volatility \(\hat{\sigma}\) under the quanto measure. Figure 2 plots the implied volatilities corresponding to the 3 year quanto call options for different strikes and different volatility/FX correlation values. The graphs confirm that a change in the correlation \(\rho_{v,FX}\) results in an almost parallel shift of the implied volatilities.

Figure 1: Correlation between VIX index and US Dollar/Euro

In order to gain an intuition for these observations we consider a Euro based trader who sold a Euro quanto option on the S&P 500 and is delta hedging the option position using a standard S&P 500 future and vega hedging using a standard variance swap or a VIX future. While all the hedge instruments are traded in US Dollar, the quanto option has no direct exposure to the US Dollar. Thus, the trader will need to setup a FX hedge in order to hedge the US Dollar exposure coming from the hedge instruments. If the S&P 500 index or the volatility change, the US Dollar value of the hedge

\(^5\) In this paper the prices of options are always expressed as a percentage of the underlying spot as it is common in the OTC market.
instruments changes which will cause the trader to dynamically rehedge the forex exposure depending on the movements of the underlying and its volatility. Consequently, the equity/forex correlation $\rho_{S,FX}$ together with the equity volatility and the forex volatility influence the trader's hedge due to the interaction of the delta hedge and the forex hedge. However, also the volatility/forex correlation $\rho_{v,FX}$ together with the volatility of volatility and the forex volatility impact the hedge result in a similar way due to the interaction of the vega hedge and the forex hedge. The first effect is well-known and taken into account by the quanto adjustment in the drift of the underlying when pricing quanto options in the standard framework. The second effect should not be ignored especially in the presence of a persistent positive volatility/forex correlation but it is only taken into account by the additional quanto adjustment in the volatility - a term which is absent in the classical quanto option pricing framework.

Table 1: Model parameters

<table>
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<tr>
<th>$\nu_0$</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\delta$</th>
<th>$\rho_{S,v}$</th>
<th>$\sigma_{FX}$</th>
<th>$\rho_{S,FX}$</th>
<th>$\rho_{v,FX}$</th>
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<td>0.175</td>
<td>0.103</td>
<td>0.131</td>
<td>0.187</td>
<td>-0.815</td>
<td>0.133</td>
<td>-0.63</td>
<td>0.51</td>
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Table 2: Model prices for different correlation values

<table>
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<th>$\rho_{v,FX}$</th>
<th>0.51</th>
<th>0.20</th>
<th>0.00</th>
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</thead>
<tbody>
<tr>
<td>Quanto Call Price</td>
<td>12.98</td>
<td>13.30</td>
<td>13.52</td>
</tr>
</tbody>
</table>

Figure 2: Implied volatilities for different correlation values
5. Comparison with standard methods and the impact of the FX skew

One inaccuracy of our model is that it only assumes Black-Scholes dynamics for the forex rate and thereby ignores the implied volatility skew or smile which can be observed in currency option markets. Although this simplifying assumption facilitated the derivation of closed-form solutions for quanto options, the question arises whether the volatility skew on forex rate options has a significant impact on standard quanto options on the underlying S and should be taken into account when pricing quanto options. In order to answer this question, we extend our model by introducing a stochastic volatility also for the forex rate process. Consequently, the extended model is described by the equations (1), (2) and by the following dynamics for $Z^{Y/X}$ under the foreign measure $Q^X$:

$$
dZ^{Y/X}_t = (r^X - r^Y)Z^{Y/X}_t dt + v_{FX}(t)Z^{Y/X}_t dW^X_Z(t), \quad Z^{Y/X}_0 = Z_0^{Y/X},$$
$$
dv_{FX}(t) = \kappa_{FX}(\theta_{FX} - v_{FX}(t))dt + \delta_{FX} dW^X_{v_{FX}}(t), \quad v_{FX}(0) = v_{FX,0}.
$$

We denote the extended model as double SV model and note that the correlation matrix between the Brownian motions $W^X_S$, $W^X_v$, $W^X_Z$, $W^X_{v_{FX}}$ is given by:

$$
\begin{pmatrix}
1 & \rho_{S,v} & \rho_{S,Z} & \rho_{S,v_{FX}} \\
\rho_{S,v} & 1 & \rho_{v,Z} & \rho_{v,v_{FX}} \\
\rho_{S,Z} & \rho_{v,Z} & 1 & \rho_{Z,v_{FX}} \\
\rho_{S,v_{FX}} & \rho_{v,v_{FX}} & \rho_{Z,v_{FX}} & 1
\end{pmatrix}.
$$

We reduce the dimensionality of the correlation matrix by choosing the parametric form:

$$
\rho_{S,v_{FX}} = \rho_{S,Z} \rho_{Z,v_{FX}}, \quad \rho_{v,Z} = \rho_{S,v} \rho_{S,Z}, \quad \rho_{v,v_{FX}} = \rho_{S,v} \rho_{S,Z} \rho_{Z,v_{FX}},$$

which matches correlation assumptions made by Dimitroff et al. [1] and also corresponds to the correlation parametrization with parameter $\beta=0$ used by Jäckel [5]. For the calibration of the double SV model we determine the stochastic volatility parameters of $v$ the same way we did before and in addition obtain the forex parameters $v_{FX,0}$, $\kappa_{FX}$, $\theta_{FX}$, $\delta_{FX}$ and $\rho_{Z,v_{FX}}$ by an analog calibration to standard options on the forex rate. The correlation parameter $\rho_{S,Z}$ is then set such that the model price of the quanto forward is matching quanto forward given by the market. In absence of closed-form solutions

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6 Jäckel [5] demonstrated that the specific choice of the parameter $\beta$ used in his correlation parametrization does not have a significant impact on the prices of quanto options as long as the model is always calibrated to a given quanto forward.
for the prices of the quanto forward and quanto options in the double SV model we apply standard Monte Carlo methods to compute these prices numerically and use the following equations in this context:

\[
F^q(t, T) = e^{(r^q - r^Y)T - t}E^Q\left[S(T)\frac{Z_{Y/X}^{Y/X}(T)}{Z_{Y/X}^{Y/X}(t)}\right],
\]

\[
C^q(t, T, K) = e^{-r^q(T-t)}E^Q\left[(S(T) - K)\frac{Z_{Y/X}^{Y/X}(T)}{Z_{Y/X}^{Y/X}(t)}\right].
\]

Calibrating the double SV model to the same market data as used before, we obtain the model parameters listed in Table 3.

<table>
<thead>
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<th>(v_0)</th>
<th>(\kappa)</th>
<th>(\theta)</th>
<th>(\delta)</th>
<th>(\rho_{SV})</th>
<th>(v_{FX,0})</th>
<th>(\kappa_{FX})</th>
<th>(\theta_{FX})</th>
<th>(\delta_{FX})</th>
<th>(\rho_{Z,FX})</th>
<th>(\rho_{SV})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.175</td>
<td>0.103</td>
<td>0.131</td>
<td>0.187</td>
<td>-0.815</td>
<td>0.147</td>
<td>0.547</td>
<td>0.101</td>
<td>0.092</td>
<td>-0.34</td>
<td>0.62</td>
</tr>
</tbody>
</table>

Based on the calibrated model parameters for the two stochastic volatility models we now compare the model prices for Euro quanto call options on the S&P 500 index with a maturity \(T=3\) years. We also include in our comparison option prices calculated using the following three commonly used ad-hoc methods for quanto options:

- Domestic-Forward-ATM-Quanto Black-Scholes (DFAQ BS) method
- Quanto-Forward-ATM-Quanto Black-Scholes (QFAQ BS) method
- Quanto-Forward-ATM-Quanto Stochastic Volatility (QFAQ SV) method

The DFAQ BS method is simply using Black's formula with the given quanto forward \(F^q_{market}(T)\), the discount factor belonging to the payment currency \(Y\) and with a volatility for the underlying equal to the implied volatility of the corresponding non-quanto option with the same strike \(K\) and the same maturity \(T\) as the quanto option. The QFAQ BS method only differs from the DFAQ BS method by using a quanto forward adjusted volatility for the underlying which is the implied volatility of the non-quanto option with the same maturity \(T\) but with the adjusted strike \(K_{adj} = K \times F^Y(T)/F^q_{market}(T)\) where \(F^Y(T)\) is

\[\text{See Jäckel [4] or Dimitroff et al. [1] for a simple derivation of the equations.}\]

\[\text{In the interest of brevity, we do not perform the change of measure to the domestic risk-neutral measure } Q^Y \text{ for the double SV model which would result in similar additional quanto adjustments as seen in the previous sections.}\]

\[\text{In order to rule out that calibration errors resulting from the calibration of the stochastic volatility model to the market data potentially overshadow the model comparison we use the implied volatilities induced by the stochastic volatility parameter listed in Table \ref{table:modelparams} throughout the comparison.}\]
the forward of the underlying asset under the foreign measure $Q^X$. Finally, the QFAQ SV method is using the closed-form solution for non-quanto standard options in the assumed stochastic volatility model for the underlying $S$ but replacing the forward with the quanto forward $F^q_{\text{market}}(T)$ and the discount factor with the discount factor belonging to the payment currency $Y$. All three approximations are commonly used in practice as outlined by Jäckel [5] from which we have also borrowed the notation for the three methods. Please note that all three common methods do not feature the additional quanto adjustment in the volatility.

The results are summarized in Table 4 and reveal that the three standard methods produce prices which are almost 100 basis points higher than the prices of our two stochastic volatility models. These higher prices can be explained by the lack of the quanto adjustment in the volatility which causes the standard methods to use a higher effective volatility for the pricing of quanto options. Note that the observed price differences are above the usual bid/offer spreads of less 50 basis points for quanto options in the OTC market which strongly suggests that ignoring the quanto adjustment in the volatility can lead to mispricing of quanto options. In contrast to this, the prices of our reduced stochastic volatility model of Section 2 agree well with prices of the fully fledged double SV model listed in Table 4. This indicates that the price impact of the forex implied volatility skew on standard quanto options is small and that our closed-form solutions derived in Section 3 could well be used for an efficient pricing and risk management of standard quanto options without a material loss of exactness even though only at-the-money FX implied volatilities are used.

Table 4: Prices of quanto call options.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Double SV model</th>
<th>SV model</th>
<th>DFAQ BS</th>
<th>QFAQ BS</th>
<th>QFAQ SV</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>32.44 (0.013)</td>
<td>32.46</td>
<td>32.90</td>
<td>33.10</td>
<td>33.10</td>
</tr>
<tr>
<td>80</td>
<td>25.21 (0.011)</td>
<td>25.24</td>
<td>25.75</td>
<td>26.03</td>
<td>26.03</td>
</tr>
<tr>
<td>90</td>
<td>18.66 (0.010)</td>
<td>18.70</td>
<td>19.25</td>
<td>19.60</td>
<td>19.60</td>
</tr>
<tr>
<td>100</td>
<td>12.95 (0.008)</td>
<td>12.98</td>
<td>13.54</td>
<td>13.94</td>
<td>13.94</td>
</tr>
<tr>
<td>110</td>
<td>8.26 (0.007)</td>
<td>8.30</td>
<td>8.80</td>
<td>9.20</td>
<td>9.20</td>
</tr>
<tr>
<td>120</td>
<td>4.80 (0.005)</td>
<td>4.82</td>
<td>5.22</td>
<td>5.56</td>
<td>5.56</td>
</tr>
<tr>
<td>130</td>
<td>2.57 (0.004)</td>
<td>2.58</td>
<td>2.86</td>
<td>3.07</td>
<td>3.07</td>
</tr>
</tbody>
</table>

10 Intuitively speaking, the QFAQ BS method is trying to reflect the different "moneyness" of the quanto option in comparison to the non-quanto option with the same strike caused by the different forwards.
11 The prices of options as well as the strikes are expressed as a percentage of the underlying spot in the Table 4.
12 Note that the QFAQ BS method and the QFAQ SV method yield exactly the same prices in our model setting which is due to the fact that the price functions for a non-quanto call option in the Black Scholes model and the stochastic volatility model are both homogeneous with respect to the forward and the strike.
13 The numbers in parentheses are sample standard deviations.
6. Conclusions

We have introduced a model for the pricing of quanto options which features stochastic volatility for the underlying. Closed-form pricing formulas for the quanto forward and standard quanto options have been derived for the model which facilitate a fast calibration of the model and an efficient pricing and risk management of standard quanto options without the need of using Monte Carlo methods or numerical solutions of PDEs. We found that in addition to the common quanto adjustment in the drift of the underlying a quanto adjustment in the volatility needs to be considered. The impact of this additional quanto adjustment has been studied and shown to be of significance for the prices of standard quanto options. Furthermore, we have numerically studied the accuracy of the obtained quanto option prices in the framework of a double stochastic volatility model with stochastic volatility for both the underlying and the forex process. In this study, we have observed that our stochastic volatility model only produced very small price differences in comparison to the benchmark prices of the double stochastic volatility model and has the advantage of offering closed-form solutions. In addition, three commonly used methods for the pricing of quanto options have been included in the numerical study with the observation that the standard methods produce price differences in comparison to the two stochastic volatility models which are above the usual bid/offer spreads and are due to the missing quanto adjustment in the volatility.

It is clear that the volatility changing effect of the additional quanto adjustment does not only have an impact on standard quanto options but also on exotic quanto options with high vega exposure like barrier options for instance. In the interest of brevity, we defer the analysis of exotic quanto options as well as a more extensive analysis of the impact of the forex smile or skew on quanto options to future work.
7. Appendix

Lemma 1. Let ν be a mean-reversion Ornstein-Uhlenbeck process under the measure Q, i.e.:
\[ d\nu(t) = \kappa (\theta - \nu(t)) \, dt + \sigma dW^{Q}(t), \nu(0) = \nu_{0}. \]
Furthermore, let the function y be defined as
\[ y(t, T, \nu(t)) = E^{Q} \left[ e^{-s_{1} \int_{t}^{T} \nu(u) \, du - s_{2} \int_{t}^{T} \nu(u) \, du + s_{3} \nu(T)} \right] \]
for arbitrary complex numbers s₁, s₂, s₃ and \(-s_{1} \nu(u)^{2} + s_{2} \nu(u)\) is lower bounded. Then y has the following solution:
\[ y(t, T, \nu(t)) = D(t, T, \nu(t), s_{1}, s_{2}, s_{3}) \]
where the function D is given by
\[
\begin{align*}
D(t, T, \nu(t), s_{1}, s_{2}, s_{3}) &= e^{\frac{1}{2} A(t, T, s_{1}, s_{2}) \nu(t)^{2} + B(t, T, s_{1}, s_{2}, s_{3}) \nu(t) + C(t, T, s_{1}, s_{2}, s_{3})} \\
A(t, T, s_{1}, s_{2}) &= \frac{\kappa}{\sigma^{2}} - \frac{\gamma_{1}}{\sigma^{2}} \frac{\Psi(\gamma_{1}, \gamma_{2})}{\Phi(\gamma_{1}, \gamma_{2})} \\
B(t, T, s_{1}, s_{2}, s_{3}) &= \frac{k \theta \gamma_{1} - \gamma_{2} \gamma_{3} + \gamma_{3} \Psi(\gamma_{1}, \gamma_{2})}{\sigma^{2} \gamma_{1} \Phi(\gamma_{1}, \gamma_{2})} - \frac{k \theta}{\sigma^{2}} \\
C(t, T, s_{1}, s_{2}, s_{3}) &= -\frac{1}{2} \ln \left( \Phi(\gamma_{1}, \gamma_{2}) \right) + \frac{\kappa}{2} (T - t) \\
&+ \left( \frac{\kappa^{2} \theta^{2} \gamma_{1}^{2} - \gamma_{3}^{2}}{2 \sigma^{2} \gamma_{1}^{2}} \right) \left[ \frac{\sinh(\gamma_{1} (T - t))}{\Phi(\gamma_{1}, \gamma_{2})} - \gamma_{1} (T - t) \right] \\
&+ \left( \frac{k \theta \gamma_{1} - \gamma_{2} \gamma_{3}}{\sigma^{2} \gamma_{1}^{3}} \right) \left[ \frac{\cosh(\gamma_{1} (T - t))}{\Phi(\gamma_{1}, \gamma_{2})} - 1 \right]
\end{align*}
\]
with
\[
\begin{align*}
\Psi(\gamma_{1}, \gamma_{2}) &= \sinh(\gamma_{1} (T - t)) + \gamma_{2} \cosh(\gamma_{1} (T - t)) \\
\Phi(\gamma_{1}, \gamma_{2}) &= \cosh(\gamma_{1} (T - t)) + \gamma_{2} \sinh(\gamma_{1} (T - t)) \\
\gamma_{1} &= \sqrt{2 \sigma^{2} s_{1} + \kappa^{2}} \\
\gamma_{2} &= \frac{1}{\gamma_{1}} (\kappa - 2 \sigma^{2} s_{3}) \\
\gamma_{3} &= \kappa^{2} \theta - s_{2} \sigma^{2},
\end{align*}
\]

Proof. See the appendix of Schöbel and Zhu [9].
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