Consistent Pricing of CMS and CMS Spread Options with SABR-like Distributions and Power-t Copulas

Andreas K.H. Geisselmeyer*

March, 2012

Abstract

Today, European-style CMS spread options are often priced with a bivariate normal distribution, which arises when modelling the CMS rates with two correlated Gaussian stochastic processes, or, equivalently, two normal marginals coupled with a Gaussian copula. In this paper, we relax the rather ad-hoc assumption of bivariate normally distributed CMS rates. First, we replace the normal marginal distributions with SABR-like skewed-t distributions. By means of a Power-t copula, we then derive a skewed and fat-tailed joint distribution as a good proxy to a bivariate SABR distribution, which is nonetheless simple, tractable and well-behaved. We present efficient semi-analytical pricing formulas for CMS spread options (as well as CMS swaps, caps and floors as the limiting cases) and achieve a good match of the SABR swaption, CMS and CMS spread option market. For larger spread option maturities, we find indications for possible inconsistencies in the CMS and CMS spread option market.

1 Introduction

CMS spread options are typically European-style today and therefore of a comparatively simple nature. The problem of their consistent modelling, i.e. recovering the SABR swaption smile as well as the CMS and CMS spread option market in a realistic way, is, however, still a topic of ongoing research. SABR is the standard model for pricing interest rate swaptions, cf. Hagan, Kumar, Lesniewski, and Woodward [2002], and is also the basis for CMS pricing via CMS replication, cf. Hagan [2003]. In the following, we briefly summarise the work on European-style spread options of the past decade.

Cherubini, Luciano, and Vecchiato [2001] use Archimedean copulas with marginals calibrated to historical data to give price bounds and pricing formulas for multivariate contingent claims. Dempster and Hong [2001] price spread options in a stochastic volatility framework via FFT assuming the existence of the joint characteristic function, eg. in the case when the underliers share the stochastic variance process. Carmona and Durrleman [2003] present a comprehensive survey on the available models and tools for spread option pricing with a special focus on commodity and energy markets. Alexander and Scourse [2003] use a bivariate lognormal mixture model to introduce a slight skew and semi-heavy tails in the joint distribution and to account for the correlation “frown”, i.e. the correlation skew. Bennett and Kennedy [2004] model the marginals with a mixture of lognormals and use a perturbed Gaussian copula for pricing quanto FX options. Berrahoui [2004] calculates the CMS rate distributions via call spreads thus taking into account the volatility smile and further applies a Gaussian copula calibrated to historical data. Benhamou and Croissant [2007] apply a Gaussian copula to marginals obtained via SABR local time approximations.

*Unicredit Group, Quantitative Product Group, Rates Quants, Arabellastrasse 12, D-81925 Munich, Germany, andreas.geisselmeyer@unicreditgroup.de.

I would like to thank Dongning Qu for suggesting this project and many helpful comments. I would also like to thank Robert Brand, Stephane Capet, Daniel Dreher, Ricardo Rueda-Nagel and Rolando Santambrogio for many insightful discussions.
Shaw and Lee [2007] and the references therein tackle Student-t copulas based on general multivariate t distributions.

With the financial crisis, spread option modelling more or less went back to the bivariate normal case, while fixing some details. Recently though, copulas have attracted attention again. Liebscher [2008] introduces multivariate product copulas, which in particular allow to create new asymmetric copulas out of existing copulas. Based on these results, Andersen and Piterbarg [2010] present Power copulas and apply a Power-Gaussian copula to CMS spread option pricing, thus generating a skew in the joint distribution. Similarly, Austing [2011] and Elices and Fouque [2012] give constructions of skewed joint distributions, the former in the FX context via best-of options, the latter via a perturbed Gaussian copula derived with asymptotic expansion techniques from the transition probabilities of a stochastic volatility model. Piterbarg [2011] derives necessary and sufficient conditions for the existence of a joint distribution consistent with vanilla and exotic prices in a CMS spread, FX cross-rate or equity basket option context. McCloud [2011] shows the dislocation of the CMS and CMS spread option market in the recent past and detects it with bounds from a copula based approach. Kienitz [2011] presents an extrapolation method for numerically obtained SABR probability distributions and applies Markovian projection to a bivariate SABR model.

To date, to the best of our knowledge, there is no model in the spread option literature that takes into account the important features of a bivariate SABR distribution, ie. skew and fat tails in the joint probability distribution. Often, light-tailed distributions such as the bivariate normal, or semi-heavy-tailed and slightly skewed distributions such as lognormal mixtures are in use. Numerically-obtained SABR marginal distributions are not very tractable (issues with efficiency/accuracy/interpolation/extrapolation) and usually coupled with simple copulas such as the Gaussian copula which leads to unnatural joint distributions.

The aim of this paper is to construct a CMS rate joint distribution with fat tails and skew which is close to a bivariate SABR distribution, but still manageable and well-behaved. We achieve this with a copula approach. We use the SABR-like skewed-t distribution to get a grip at the SABR probability distribution. Such distributions are simple, tractable and well-behaved, and can take into account skew and fat tails of the SABR distribution. We then apply a Power-t copula which is rich enough to generate a natural joint distribution with skew and heavy tails, but simple enough to maintain tractability. From there, we present semi-analytical pricing formulas for CMS spread options (and CMS swaps, caps and floors as the limiting cases) and are able to achieve a good fit to the swaption, CMS and CMS spread option market. For larger maturities and high strikes, skewed-t marginals lead to consistently higher CMS spread cap prices than the market (due to more probability mass in the tails), which is an indication for possible inconsistencies in the CMS and CMS spread option market.

The rest of the paper is organised as follows. In Section 2, we review standard European-style CMS spread options, introduce our notation and outline the main pricing problem. In Section 3, we present the different marginal distributions we will use, from normal marginals to SABR marginals and SABR-like skewed-t marginals. We apply skewed-t distributions for the first time in the spread option literature. In Section 4, we recall some known and less-known results on copulas and review Gaussian, Power-Gaussian and t copulas. We then present the Power-t copula as a special case of Power copulas, which has not been used in the spread option literature before. Section 5 presents the pricing formulas for CMS caps and CMS spread caps (with CMS floors, CMS spread floors and CMS swaps following immediately). Section 6 gives numerical results. We show calibrations for the Gaussian copula with normal marginals (the standard bivariate normal model), the Power-Gaussian copula with normal marginals and the Power-t copula with skewed-t marginals. We further analyse how the different choices of marginals and copulas affect the prices of CMS caps, CMS spread digitals and CMS spread options. Section 7 concludes.
2 The Pricing Problem

Let $S_1$ and $S_2$ be two CMS rates with different tenors, e.g. 10y and 2y, and $0 = T_0 < T_1 < \cdots < T_{n+1}$ a set of equally-spaced dates. Standard CMS spread options are either caps or floors and thus a sum of CMS spread caplets/floorlets with payoffs

$$h(S_1(T_i), S_2(T_i)) = \tau_i \max(\omega(S_1(T_i) - S_2(T_i) - K), 0), \quad i = 1, \ldots, n$$

where $\omega = \pm 1$ and $K$ a fixed strike. $S_1$ and $S_2$ are fixed at $T_i$, payment of $h(S_1(T_i), S_2(T_i))$ usually occurs at $T_{i+1}$, $\tau_i = T_{i+1} - T_i$ is a year fraction, $i = 1, \ldots, n$.

The present value of a CMS spread cap/floor with maturity $T_{n+1}$ is

$$\sum_{i=1}^{n} \tau_i P(0, T_{i+1}) \mathbb{E}^{T_{i+1}} \left[ \max(\omega(S_1(T_i) - S_2(T_i) - K), 0) \right].$$

Expectations are taken under the $T_{i+1}$-forward measures. In the Euro area, fixing/payment dates are typically 3m-spaced. Note that the first caplet/floorlet which is fixed at $T_0$ is excluded. When $S_2$ is removed from the payoff, we arrive at a CMS cap/floor, when we replace $S_2$ with a 3m-Euribor, we obtain a CMS swap. Our pricing problem is therefore of the form

$$V = \mathbb{E}^{T_p} \left[ \max(\omega(S_1(T) - S_2(T) - K), 0) \right]$$

with $T_p \geq T$ and $\psi(s_1, s_2)$ the joint probability density of $S_1(T)$ and $S_2(T)$.

In the following, we will work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{ \mathcal{F}_t \}_{t \geq 0}$ denotes a filtration of $\mathcal{F}$ satisfying the usual conditions. We assume the existence of equivalent martingale measures $\mathbb{Q}$ and write $\mathbb{E}^{\mathbb{Q}}[\cdot]$ for the expected value w.r.t. such measures. We assume that the payoff $h(S_1(T), S_2(T))$ is $\mathcal{F}_T$-measurable and satisfies the necessary integrability conditions.

For a bivariate normal distribution, the joint density function is available in closed form. Otherwise, this is rarely the case. We can, however, always represent a joint density in terms of its marginals and a (unique) copula (Sklar’s theorem). Let $\Psi(s_1, s_2)$ be the joint cumulative distribution function (cdf), $\Psi_i(s_i), i = 1, 2$, the marginal cdfs, $\psi_i(s_i), i = 1, 2$, the marginal densities and $C(u_1, u_2)$ a copula. Then

$$\Psi(s_1, s_2) = C(\Psi_1(s_1), \Psi_2(s_2))$$

$$\psi(s_1, s_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(\Psi_1(s_1), \Psi_2(s_2)) \psi_1(s_1) \psi_2(s_2)$$

and it follows that

$$V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s_1, s_2) \frac{\partial^2}{\partial u_1 \partial u_2} C(\Psi_1(s_1), \Psi_2(s_2)) \psi_1(s_1) \psi_2(s_2) ds_1 ds_2.$$ 

If we can represent the above 2d integral as a 1d integral, pricing becomes efficient.

3 Marginal Distributions

In this section, we present the different marginal distributions which we will use, from normal marginals to SABR marginals and SABR-like skewed-t marginals. Normal marginals will be used together with a Gaussian copula (the standard model), but also in the Power-Gaussian copula extension. We present the main properties (and peculiarities) of the (numerical) SABR distribution and finally the SABR-like skewed-t distribution, which will be applied in a Power-t copula setting.
3.1 Gaussian Marginals

We assume that the CMS rates $S_1$ and $S_2$ follow Gaussian processes:

\[ dS_i(t) = \sigma_i dW_i^T, \quad S_i(0) = \mathbb{E}_T^p[S_i(T)], \quad i = 1, 2 \]

$W^T$ denotes a standard Wiener process w.r.t. the $T_p$-forward measure. The volatilities $\sigma_i$ are chosen constant. Naturally, CMS rates are martingales under their respective annuity measure. We can assume no drifts under the $T_p$-forward measure by using the convexity-adjust CMS rate $S_i(0) = \mathbb{E}_T^p[S_i(T)]$ with $T$ the fixing date.

3.2 SABR Marginals

We assume that the CMS rates $S_1$ and $S_2$ follow SABR processes:

\[ dS_i(t) = \alpha_i(t)S_i(t)^{\beta_i}dW_i^{A_i(T)}, \quad S_i(0) = S_{i,0} \]
\[ d\alpha_i(t) = \nu_i \alpha_i(t) dZ_i^{A_i(T)}, \quad \alpha_i(0) = \alpha_i, \quad i = 1, 2 \]
\[ d(W_i, Z_i) = \rho_i dt \]

$\alpha_i$ determines the swaption volatility level, $\beta_i$ and $\rho_i$ the skew and $\nu_i$ the volatility smile. $W^{A_i(T)}$ and $Z^{A_i(T)}$ denote standard Wiener processes w.r.t. the $A_i(T)$-annuity measure. To obtain marginal SABR densities $\psi_i^{A_i(T)}(\cdot)$ and cdfs $\Psi_i^{A_i(T)}(\cdot)$ under the $T_p$-forward measure, we either use CMS digitals (caplet spreads) obtained via CMS replication, cf. Hagan [2003],

\[ \psi_i^{T_p}(x) = 1 + \frac{\partial E^{T_p}(\max(S_i(T) - x, 0))}{\partial x} \]
\[ \psi_i^{T_p}(x) = \frac{\partial^2 E^{T_p}(\max(S_i(T) - x, 0))}{\partial x^2} \]

or apply the above formulas to payer swaption spreads to obtain $\psi_i^{A_i(T)}(\cdot)$ and then relate, cf. Andersen and Piterbarg [2010], Section 16.6.9:

\[ \psi_i^{T_p}(x) = \int_{-\infty}^{x} \frac{G_i(u, T_p)}{G_i(S_{i,0}, T_p)} \psi_i^{A_i(T)}(u) du \]
\[ \psi_i^{T_p}(x) = \frac{G_i(x, T_p)}{G_i(S_{i,0}, T_p)} \psi_i^{A_i(T)}(x). \]

$G$ arises from the change of measure from the $A_i(T)$-annuity measure to the $T_p$-forward measure and approximates the quotient $\frac{P_i(T_p)}{A_i(T)}$ via a yield curve model, cf. Hagan [2003]. We find that both approaches produce nearly identical $T_p$-forward measure densities and cdfs. This implies that when we are able to recover CMS caplet prices (from CMS replication) with a density $\psi^*$, we also recover SABR swaption prices with the density transform from above (and vice versa).

Although it is desirable to use SABR marginals for spread option pricing, there are a few issues that complicate matters, cf. Andersen and Piterbarg [2005], Henry-Labordre [2008], Jourdain [2004]:

- $\beta = 1$: For $\rho > 0$, $S$ is not a martingale (explosion).
- $\beta \in (0, 1)$: $S = 0$ is an attainable boundary that either is absorbing ($\frac{1}{2} \leq \beta < 1$) or has to be chosen absorbing ($0 < \beta < \frac{1}{2}$) to avoid arbitrage.\footnote{The arbitrage opportunity that arises when choosing a reflecting boundary is quite theoretical. $S$ is, however, not a martingale anymore.}
- $\beta = 0$: $S$ can become negative.

Usually, the SABR model behaves fairly well around at-the-money. We now have a brief look at what happens far-from-the-money (low strikes and high strikes).
Low Strikes:

- \( \beta = 1 \): This case is not really relevant for interest rate modelling unless \( \nu \) is set to 0 (lognormal model).
- \( \beta \in (0, 1) \): This case is typical for the Euro markets. For low forwards and high volatilities, the probability of absorption can be quite high which in terms of financial modelling is unpleasant.

The SABR density is not available in (semi-)analytical form. The standard volatility formula from Hagan, Kumar, Lesniewski, and Woodward [2002] applied to payer swaption spreads leads to negative probabilities when \( \nu^2T \ll 1 \) does not hold (the singular perturbation techniques fail), the density formula from Hagan, Lesniewski, and Woodward [2005] behaves better but also gets less accurate the higher \( \nu^2T \) is (the asymptotic expansions fail) and Monte Carlo is converging too slowly.

- \( \beta = 0 \): This can be an interesting alternative in low interest rate environments. The density is purely diffusive.

High Strikes: Generally, it is problematic to apply a model far away from the strike region it has been calibrated to. In case of the SABR model, the missing mean-reverting drift also leads to too high volatilities in the right wing.

3.3 Skewed-t Marginals

Due to the many issues we face in the relevant case \( \beta \in (0, 1) \), we will not use numerically computed SABR distributions directly. We instead resort to mapping the important features of the SABR distribution (location, scale, skew and kurtosis/tail behaviour) to a well-behaved and tractable distribution: the skewed-t distribution. This way, we also solve the problem of interpolation/extrapolation. As long as the numerical SABR distribution is not too distorted \( (\nu^2T \ll 1) \), we will achieve a surprisingly good fit. When the SABR volatility formula starts to induce negative probabilities, the fitting quality will of course deteriorate. Skewed-t distributions have not been used in the spread option literature before.

We derive the skewed-t distribution in steps starting with the Student-t distribution. The Student-t distribution with \( d \) degrees of freedom has the density

\[
  f_d(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\sqrt{\pi d}} \left(1 + \frac{x^2}{d}\right)^{-\frac{d+1}{2}}
\]

where \( \Gamma \) denotes the Gamma function. Introducing location and scale parameters \( \mu \) and \( \sigma \), we get

\[
  f_{\mu,\sigma,d}(x) = \frac{1}{\sigma} f_d\left(\frac{x - \mu}{\sigma}\right).
\]

Introducing a skew parameter \( \varepsilon \), cf. Fernandez and Steel [1996], leads to

\[
  f_{\varepsilon,d}(x) = \frac{2}{\varepsilon + \frac{1}{2}} \left(f_d\left(\varepsilon x\right) \mathbb{1}_{x<0} + f_d\left(\frac{x}{\varepsilon}\right) \mathbb{1}_{x>0}\right).
\]

Combining, we arrive at the skewed-t distribution with low density, cdf and inverse cdf

\[
  f_{\mu,\sigma,\varepsilon,d}(x) = \frac{2}{\varepsilon (\varepsilon + 1)} f_d\left(\frac{\varepsilon(x - \mu)}{\sigma}\right) \mathbb{1}_{x<\mu} + \frac{2}{\varepsilon (\varepsilon + 1)} f_d\left(\frac{x - \mu}{\varepsilon \sigma}\right) \mathbb{1}_{x\geq \mu},
\]

\[
  F_{\mu,\sigma,\varepsilon,d}(x) = \frac{2}{1 + \varepsilon^2} F_d\left(\frac{\varepsilon(x - \mu)}{\sigma}\right) \mathbb{1}_{x<\mu} + \frac{2}{1 + \varepsilon^2} F_d\left(\frac{x - \mu}{\varepsilon \sigma}\right) \mathbb{1}_{x\geq \mu},
\]

\[
  F^{-1}_{\mu,\sigma,\varepsilon,d}(x) = \mu + \frac{\sigma}{\varepsilon} F^{-1}_d\left(\frac{x(1 + \varepsilon^2)}{2}\right) \mathbb{1}_{x<\frac{1}{1 + \varepsilon^2}} - \sigma \varepsilon F^{-1}_d\left(\frac{(1 - x)(1 + \frac{1}{\varepsilon^2})}{2}\right) \mathbb{1}_{x\geq \frac{1}{1 + \varepsilon^2}}.
\]
$F_d$ and $F_d^{-1}$ denote cdf and inverse cdf of the Student-t distribution. With $\varepsilon = 1$ and $d = \infty$, we recover the normal distribution with mean $\mu$ and standard deviation $\sigma$. With $\mu = 0$, $\sigma = 1$ and $\varepsilon = 1$, we recover the Student-t distribution.

$\varepsilon < 1$ produces a negative skew, $\varepsilon > 1$ a positive skew, cf. Figure 1. We see that with a negative/positive skew, the expected value is left/right of the peak respectively. Note that the expected value does not coincide with parameter $\mu$ when $\varepsilon \neq 1$.

In the literature the name skewed-t distribution is also used for other fat-tailed distributions, eg. special cases of the generalised hyperbolic distribution. The advantage of the above definition is that it naturally extends the normal and Student-t distribution for which efficient implementations exist.

We end this section with some numerical results in order to better understand how SABR densities can be computed numerically and how the skewed-t distribution can be calibrated. We calculated (annuity measure) SABR densities via

- a Monte Carlo simulation of the SABR SDE using an Euler discretization with 100000 paths and 300 time steps per year.
- the second derivative of European payer swaptions, cf. Section 3.2, applying the volatility formula from Hagan, Kumar, Lesniewski, and Woodward [2002]. We will call this approach 'Hagan'.
- the probability density function from Hagan, Lesniewski, and Woodward [2005]. We will call this method 'Lesniewski'.

Market data was based on Totem (cf. Section 6). Selected Totem-implied SABR $\nu$’s are given in Table 1. Plots for the calculated SABR densities are given in Figures 2 and 3. We calibrated the skewed-t distribution to both the Monte Carlo density and the density based on Hagan’s volatility formula. We find that the skewed-t distribution matches the Monte Carlo density very well. With increasing option maturity $T$, the degree-of-freedom parameter decreases, ie. the tail-thickness increases. We further see that with growing $\nu^2T$, Hagan’s volatility formula leads to densities which deviate more and more from Monte Carlo and eventually have negative probabilities around strike 0. It then also becomes harder to match a skewed-t distribution. We observe that when fitting the skewed-t distribution to Hagan, the degree-of-freedom parameter is significantly lower.
compared to a fitting to the Monte Carlo density. This indicates that Hagan’s formula blows up the second moment and hence the convexity-adjusted CMS rates, cf. [Andersen and Piterbarg 2005]. Lesniewski’s probability formula performs well, but is too symmetric and also deviates from Monte Carlo with increasing $\nu^2T$. Since in practise typically Hagan’s formula is used, we will not consider this method here any further.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$F$</th>
<th>$\nu$</th>
<th>$\nu^2T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2y</td>
<td>3.83%</td>
<td>46.60%</td>
<td>0.43</td>
</tr>
<tr>
<td>5y</td>
<td>4.25%</td>
<td>45.90%</td>
<td>1.05</td>
</tr>
<tr>
<td>10y</td>
<td>4.35%</td>
<td>40.39%</td>
<td>1.63</td>
</tr>
</tbody>
</table>

Table 1: SABR $\nu$’s for 10y CMS rates with different maturities.

4 Copulas

In Section 2 we have already stated how copulas can be applied to spread option pricing. We now present the copula families that we will use in the following. Very good references for copulas are Cherubini, Luciano, and Vecchiato [2004] and Nelsen [2006], a comprehensive reference for multivariate t distributions can be found in Kotz and Nadarajah [2004].

We give formulas for the (light tail) Gaussian copula which has commonly been used (and abused) in finance, define the (fat tail) t copula and extend both copulas in terms of a Power copula to generate skewness in the joint probability distribution.

Power copulas were presented in [Andersen and Piterbarg 2010] as special cases of product copulas. Product copulas were introduced by Liebscher [2008] and allow to create new asymmetric copulas out of existing symmetric ones. Andersen and Piterbarg [2010] applied a Power-Gaussian copula to CMS spread option pricing in order to recover the correlation skew. Power-t copulas have not been used in the spread option literature before.

We end the section with an overview of different combinations of marginals and copulas and that way motivate our test cases for Section 6.

Gaussian copulas are given by

$$C_G(u_1, u_2) = \Phi_\rho(\Phi^{-1}(u_1), \Phi^{-1}(u_2)),$$

where $\Phi_\rho$ denotes the bivariate standard normal cumulative distribution function (cdf) with correlation $\rho$ and $\Phi$ the univariate standard normal cdf. We define t copulas as

$$C_T(u_1, u_2) = F_{\rho,d}(F_d^{-1}(u_1), F_d^{-1}(u_2)),$$

where $F_{\rho,d}$ denotes the bivariate Student-t cdf with correlation $\rho$ and degrees of freedom $d$. $F_d$ is the univariate Student-t cdf with degrees of freedom $d$, cf. Section 3.3 It holds:

$$F_{\rho,d}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\rho^2}} \left( 1 + \frac{s^2 + t^2 - 2\rho st}{d(1-\rho^2)} \right)^{-\frac{d+2}{2}} ds dt.$$

With $d = \infty$ we get back to a Gaussian copula. Power-Gaussian copulas, cf. [Andersen and Piterbarg 2010], are given by

$$C_{PG}(u_1, u_2) = u_1^{-\theta_1}u_2^{-\theta_2}C_G(u_1^{\theta_1}, u_2^{\theta_2}).$$

The parameters $\theta_1$ and $\theta_2$ are defined in $[0,1]$ and generate a skew in the joint distribution. $\theta_1 = \theta_2 = 1$ yields the Gaussian copula again. Finally, we introduce Power-t copulas:

$$C_{PT}(u_1, u_2) = u_1^{-\theta_1}u_2^{-\theta_2}C_T(u_1^{\theta_1}, u_2^{\theta_2}).$$
Figure 2: SABR densities of 10y CMS rates with different maturities. Skewed-t distribution calibrated to MC.
Figure 3: SABR densities of 10y CMS rates with different maturities. Skewed-t distribution calibrated to Hagan.
The Power-t copula family comprises Power-Gaussian copulas \((d = \infty)\) as well. For spread option pricing and joint density plots, we need some partial derivatives and technical properties of copulas which are given in Appendix A.1.

We will now have a look at the joint distributions that are generated by different combinations of marginals and copulas by applying the joint density formula

\[
\psi(s_1, s_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(\Psi_1(s_1), \Psi_2(s_2)) \psi_1(s_1) \psi_2(s_2).
\]

To keep things simple, we do not consider skews for the moment, i.e., we combine (light tail) normal and (fat tail) Student-t marginals with (light tail) Gaussian and (fat tail) t copulas. Contour plots of the resulting joint densities are shown in Figure 4. Darker colours mean higher density values, for really low density values, we plotted black contour lines to better illustrate the tail behaviour.

We clearly see that Gaussian copulas with normal marginals and t copulas with t marginals generate natural joint densities. Comparing these two combinations, we observe that in the normal case, probability mass is quite concentrated around the center, while in the t case, there is more mass in the tails which has important pricing consequences.

Cross combinations of marginals and copulas are shown in the remaining two plots of Figure 4. Such combinations typically lead to distorted joint densities. When t marginals are used with a Gaussian copula, probability mass is shifted to the corners. When normal marginals are applied to a t copula, probability mass gets more concentrated in the center.

Spread option pricing tests show that for high strikes t marginals lead to higher spread cap prices than normal marginals, while t copulas lead to lower prices than Gaussian copulas. As a rule of thumb, the smallest prices are generated by a t copula with normal marginals, while the largest prices are generated by a Gaussian copula with t marginals.

In Section 6, we will therefore focus on extensions of the natural cases: Power-Gaussian copulas with normal marginals and Power-t copulas with skewed-t marginals.

5 Pricing Formulas

In this section, we present the pricing formulas for CMS caplets and CMS spread caplets. Prices for CMS floorlets and CMS spread floorlets follow immediately from put-call parity. CMS digitals can be calculated via caplet spreads, cf. Section 3.2. Convexity-adjusted CMS rates are calculated as CMS caplets with strike 0 and once we can calculate convexity-adjusted CMS rates, we can also price CMS swaplets. This section is rather technical and the reader might as well directly skip to Section 6.

5.1 CMS Caplets

We first have a look at the different formulas for CMS caplets. In case of normal marginals, we simply use Bachelier’s formula. In the case of SABR marginals, we price CMS caplets with CMS replication. The CMS caplet formula for skewed-t marginals is new and will enable us to calculate convexity-adjusted CMS rates and CMS caplets close to CMS replication.

**Lemma 5.1:** When, under the \(T_p\)-forward measure, the CMS rate \(S(T)\) is normally distributed with mean \(S(0) = \mathbb{E}^{T_p}[S(T)]\) and volatility \(\sigma\), the undiscounted price of a CMS caplet with maturity \(T\), payment date \(T_p\) and strike \(K \geq 0\) is given by

\[
V = (S(0) - K) \Phi \left( \frac{S(0) - K}{\sigma \sqrt{T_p}} \right) + \sigma \sqrt{T} \varphi \left( \frac{S(0) - K}{\sigma \sqrt{T}} \right).
\]
Figure 4: Joint densities for different combinations of marginals and copulas.
Lemma 5.2: When, under the $T_p$-forward measure, the CMS rate $S(T)$ follows a skewed-t distribution with density $f_{\mu, \sigma, \epsilon, d}$, the undiscounted price of a CMS caplet with maturity $T$, payment date $T_p$ and strike $K \geq 0$ is given by

$$V = \begin{cases} \mu + 2\sigma \left( \frac{1}{\epsilon} \frac{d}{d-1} f_d(0) - K + \frac{2}{\epsilon + \frac{1}{\epsilon}} \left( \frac{\sigma^2 + \mu^2}{\epsilon \sigma} f_d \left( \frac{(K-\mu)}{\sigma} \right) + \frac{K-\mu}{\epsilon} F_d \left( \frac{(K-\mu)}{\sigma} \right) \right) \right) & K < \mu \\ \frac{2}{\epsilon + \frac{1}{\epsilon}} \left( \frac{\epsilon}{d-1} f_d \left( \frac{K-\mu}{\sigma \epsilon} \right) - \epsilon (K-\mu) \left( 1 - F_d \left( \frac{K-\mu}{\epsilon \sigma} \right) \right) \right) & K \geq \mu. \end{cases}$$

For efficient pricing, the general 2d pricing integral needs to be reduced to 1d integrals.

$$S(0) = \mathbb{E}^{T_p}[S(T)] = \mu + 2\sigma \left( \frac{d}{d-1} f_d(0) + \frac{2}{\epsilon + \frac{1}{\epsilon}} \left( \frac{\sigma^2 + \mu^2}{\epsilon \sigma} f_d \left( \frac{\epsilon \mu}{\sigma} \right) - \frac{\epsilon \mu}{\sigma} F_d \left( \frac{\epsilon \mu}{\sigma} \right) \right) \right)$$

Proof. Appendix A.2. \hfill \Box

5.2 CMS Spread Caplets

We continue with the CMS spread option pricing formula in a bivariate normal model and then review the general 1d copula spread option pricing formula which can be applied to all marginal and copula combinations.

Lemma 5.3: When, under the $T_p$-forward measure, CMS rates $S_1(T)$ and $S_2(T)$ follow a bivariate normal distribution with means $S_1(0) = \mathbb{E}^{T_p}[S_1(T)]$ and $S_2(0) = \mathbb{E}^{T_p}[S_2(T)]$, volatilities $\sigma_1$ and $\sigma_2$ and correlation $\rho_N$, the undiscounted price of a CMS spread caplet with maturity $T$, payment date $T_p$ and strike $K$ is given by

$$V = v(d\Phi(d) + \phi(d))$$

$$v = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_N \sigma_1 \sigma_2 \sqrt{T}}$$

$$d = \omega \frac{S_1(0) - S_2(0) - K}{v}$$

The above price can equivalently be obtained via a Gaussian copula with normal marginals. For efficient pricing, the general 2d pricing integral

$$V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s_1, s_2) \psi(s_1, s_2) ds_1 ds_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s_1, s_2) \frac{\partial^2}{\partial u_1 \partial u_2} C(\Psi_1(s_1), \Psi_2(s_2)) \psi_1(s_1) \psi_2(s_2) ds_1 ds_2$$

needs to be reduced to 1d integrals.

Lemma 5.4: When, under the $T_p$-forward measure, CMS rates $S_1(T)$ and $S_2(T)$ have marginal cdfs $\Psi_i$ and densities $\psi_i$, $i = 1, 2$, coupled with a copula $C$, the undiscounted price of a CMS spread caplet with maturity $T$, payment date $T_p$ and strike $K$ is given by

$$V = \int_{-\infty}^{\infty} (s_1 - K) \frac{\partial}{\partial u_1} C(\Psi_1(s_1), \Psi_2(s_1 - K)) \psi_1(s_1) ds_1$$

$$- \int_{-\infty}^{\infty} s_2 \left( 1 - \frac{\partial}{\partial u_2} C(\Psi_1(s_2 + K), \Psi_2(s)) \right) \psi_2(s_2) ds_2.$$

Proof. Appendix A.3. \hfill \Box

6 Numerical Results

We now apply the formulas from Sections 3 to 5 and analyse how different marginals and copulas affect prices of CMS caplets, CMS spread digitals and CMS spread caplets. We will examine in detail the combinations.
• normal marginals and Gaussian copula,
• normal marginals and Power-Gaussian copula,
• skewed-t marginals and Power-t copula.

Market data was taken from May 2011. Yield curves and SABR parameters were derived from Totem consensus\(^2\) CMS spread option quotes were taken from ICAP, cf. Table 2. In the following, \(S_1\) will denote the 10y, \(S_2\) the 2y CMS rate. All prices will be stated with discount factor and year fraction dropped.

<table>
<thead>
<tr>
<th>(T) (K)</th>
<th>Flr</th>
<th>Flr</th>
<th>Flr</th>
<th>Cap</th>
<th>Cap</th>
<th>Cap</th>
<th>Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1y</td>
<td>0.6</td>
<td>0.7</td>
<td>0.9</td>
<td>66.7</td>
<td>48.7</td>
<td>31.9</td>
<td>17.6</td>
</tr>
<tr>
<td>2y</td>
<td>5.7</td>
<td>6.7</td>
<td>7.5</td>
<td>145.7</td>
<td>106.8</td>
<td>71.2</td>
<td>42.0</td>
</tr>
<tr>
<td>3y</td>
<td>12.4</td>
<td>14.9</td>
<td>17.0</td>
<td>210.8</td>
<td>154.1</td>
<td>103.6</td>
<td>63.0</td>
</tr>
<tr>
<td>4y</td>
<td>22.7</td>
<td>27.2</td>
<td>30.9</td>
<td>267.7</td>
<td>195.1</td>
<td>131.9</td>
<td>82.0</td>
</tr>
<tr>
<td>5y</td>
<td>36.1</td>
<td>42.8</td>
<td>48.3</td>
<td>322.3</td>
<td>235.0</td>
<td>160.1</td>
<td>101.6</td>
</tr>
<tr>
<td>7y</td>
<td>71.0</td>
<td>82.8</td>
<td>92.1</td>
<td>434.9</td>
<td>320.5</td>
<td>223.5</td>
<td>148.2</td>
</tr>
<tr>
<td>10y</td>
<td>143.0</td>
<td>162.2</td>
<td>177.3</td>
<td>605.8</td>
<td>455.2</td>
<td>328.7</td>
<td>230.5</td>
</tr>
</tbody>
</table>

Table 2: CMS spread option quotes (in bps) from ICAP.

6.1 The Standard Model: Normal Marginals and Gaussian Copula

Normal marginals were calibrated to the swaption volatility skew, the calibrated parameters are given in Table 3. Gaussian copula correlations were obtained by a calibration to the spread option quotes from Table 2 using Lemma 5.3. For each spread option maturity and strike one correlation was obtained, such that the market quote was perfectly matched. The resulting strike-dependent normal correlations \(\rho_N(K)\) for maturities \(T = 2y, T = 5y\) and \(T = 10y\) are given in Table 4. Similarly to strike-dependent Black-Scholes volatilities, strike-dependent correlations are an ad-hoc fix for a too simplistic model. This is called the correlation skew.

<table>
<thead>
<tr>
<th>(T) (K)</th>
<th>Flr</th>
<th>Flr</th>
<th>Flr</th>
<th>Cap</th>
<th>Cap</th>
<th>Cap</th>
<th>Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>2y</td>
<td>3.92%</td>
<td>3.06%</td>
<td>0.89%</td>
<td>0.97%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5y</td>
<td>4.51%</td>
<td>3.92%</td>
<td>0.84%</td>
<td>0.89%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10y</td>
<td>4.92%</td>
<td>4.60%</td>
<td>0.75%</td>
<td>0.78%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Calibrated normal marginal parameters.

<table>
<thead>
<tr>
<th>(T) (K)</th>
<th>Flr</th>
<th>Flr</th>
<th>Flr</th>
<th>Cap</th>
<th>Cap</th>
<th>Cap</th>
<th>Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>2y</td>
<td>73.4%</td>
<td>75.8%</td>
<td>77.6%</td>
<td>84.4%</td>
<td>83.8%</td>
<td>84.1%</td>
<td>85.3%</td>
</tr>
<tr>
<td>5y</td>
<td>78.9%</td>
<td>80.4%</td>
<td>81.4%</td>
<td>83.7%</td>
<td>84.1%</td>
<td>84.6%</td>
<td>84.9%</td>
</tr>
<tr>
<td>10y</td>
<td>75.0%</td>
<td>76.4%</td>
<td>77.0%</td>
<td>76.4%</td>
<td>77.0%</td>
<td>77.2%</td>
<td>76.4%</td>
</tr>
</tbody>
</table>

Table 4: Calibrated strike-dependent normal correlations for 10y2y spread caplets.

---

\(^2\)The Markit Totem service is a service that provides financial institutions with consensus prices to check their trading book valuations. All major banks participate. Every month end, a predefined set of prices has to be contributed. When a participating bank meets the required accuracy requirements, it receives the consensus prices (averaged across the participants), otherwise, it does not.
The calibrated strike-dependent normal correlations from Table 4 are fairly smooth, but also show a small jump at strike 0.25%. This strike marks the transition from floor to cap quotes and can easily become misaligned. When we interpolate over such jumps, CMS spread digitals priced via caplet spreads become distorted, cf. Figure 5. Calibrated correlations can easily become more jagged and eventually generate digital prices outside of [0, 1].

We further priced a range of CMS caplets under the normal assumption, cf. Lemma 5.1. Comparing to CMS caplets priced with CMS replication, we observe that normal CMS caplet prices decay to 0 too rapidly as a consequence of almost no probability mass in the tails of the normal distribution, cf. Figure 6.

6.2 The Extended Model: Normal Marginals and Power-Gaussian Copula

In this section, we replace the Gaussian copula from the last section with a Power-Gaussian copula to see what we can improve. We calibrated the Power-Gaussian copula directly to the normal correlations from Table 4, using Lemmas 5.3 and 5.4. The resulting copula parameters are given in Table 5. The calibration fit in terms of normal correlations is given in Figure 8. We can see that the Power-Gaussian copula calibrates very well and smooths out problematic points. As a result, CMS spread digitals priced via caplet spreads are also smooth, cf. Figure 7. CMS caplet prices remain the same as in the last section since we did not change the marginals. The Power-Gaussian copula can therefore be viewed as a neat interpolation/extrapolation method for the strike-dependent correlations that also irons out problematic correlations values.

To better understand the impact of each copula parameters, we shifted the parameters by 1% up and down. For each maturity, a different parameter sensitivity is shown in Figure 8. In terms of implied correlations,

- shifting $\rho$ up leads to a shift up
Figure 6: Comparison of 10y and 2y CMS caplets (in bps) priced with both a normal distribution and CMS replication.

Figure 7: 10y2y spread digitals priced with a Power-Gaussian copula and normal marginals.
• shifting $\theta_1$ up leads to a counterclockwise rotation with pivot at the left
• shifting $\theta_2$ up leads to a clockwise rotation with pivot at the right
• shifting $\rho$ up and $\theta_1$ and $\theta_2$ down increases the curvature.

We end this section with an important last remark: notably, Gaussian and Power-Gaussian copulas with normal marginals calibrate very well to spread option quotes. This is evidence that the market uses indeed a (light tail) normal model for CMS spread options.

6.3 The New Model: Skewed-t Marginals and Power-t Copula

Using Lemma 5.2, we first calibrated the skewed-t marginals to convexity-adjusted CMS rates and CMS digitals obtained via CMS replication. The calibration fit to CMS digitals was very good and is shown in Figure 11. For comparison, we also fitted normal distributions which tend to 0 and 1 much faster than the skewed-t distributions. Calibrated skewed-t parameters are shown in Tables 6 and 7. With these parameters, we perfectly match convexity adjusted CMS rates and CMS caplets, cf. Table 8 and Figure 9.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\varepsilon$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2y</td>
<td>0.0392</td>
<td>0.0106</td>
<td>0.992</td>
<td>4.77</td>
</tr>
<tr>
<td>5y</td>
<td>0.0446</td>
<td>0.0122</td>
<td>0.974</td>
<td>1.97</td>
</tr>
<tr>
<td>10y</td>
<td>0.0470</td>
<td>0.0118</td>
<td>0.939</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Table 6: Calibrated skewed-t parameters for 10y CMS digitals.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\varepsilon$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2y</td>
<td>0.0263</td>
<td>0.0111</td>
<td>1.21</td>
<td>4.35</td>
</tr>
<tr>
<td>5y</td>
<td>0.0383</td>
<td>0.0139</td>
<td>0.982</td>
<td>2.23</td>
</tr>
<tr>
<td>10y</td>
<td>0.0480</td>
<td>0.0146</td>
<td>0.876</td>
<td>1.59</td>
</tr>
</tbody>
</table>

Table 7: Calibrated skewed-t parameters for 2y CMS digitals.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$S_1(0)$</th>
<th>$S_2(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2y</td>
<td>3.92%</td>
<td>3.06%</td>
</tr>
<tr>
<td>5y</td>
<td>4.51%</td>
<td>3.93%</td>
</tr>
<tr>
<td>10y</td>
<td>4.92%</td>
<td>4.60%</td>
</tr>
</tbody>
</table>

Table 8: Convexity-adjusted CMS rates implied by the calibrated skewed-t distributions.

We then studied the effect of a Power-t copula with skewed-t marginals on spread option prices. We calibrated the Power-t copula to the normal correlations from Table 4. We only calibrated $\rho$, $\theta_1$ and $\theta_2$. $d$ was set to the average of the marginal degree-of-freedom parameters. The resulting copula parameters are given in Table 9. The calibration fit in terms of normal correlations is given in Figure 12. We can see that the Power-t copula calibrates well for smaller maturities and not so well for larger maturities. The reason is that high-strike spread caplets have higher prices (i.e., lower correlations) in case of a fat-tailed joint distribution than in case of a bivariate normal distribution. This is the 2d analogy to our CMS caplet example where prices obtained with a normal distribution decay to 0 much quicker than prices from CMS replication/a skewed-t distribution. This is an indication for inconsistent pricing in the CMS and CMS spread option market. While CMS options are priced with CMS replication (based on SABR), CMS spread
Figure 8: Power-Gaussian copula calibrated to 10y2y normal correlations and parameter sensitivities.
Table 9: Calibrated Power-t copula parameters for 10y-2y spread caplets.

<table>
<thead>
<tr>
<th>T</th>
<th>(\rho)</th>
<th>(\theta_1)</th>
<th>(\theta_2)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2y</td>
<td>90%</td>
<td>96%</td>
<td>96%</td>
<td>4.5</td>
</tr>
<tr>
<td>5y</td>
<td>90%</td>
<td>99%</td>
<td>97.5%</td>
<td>2.1</td>
</tr>
<tr>
<td>10y</td>
<td>92%</td>
<td>99%</td>
<td>95%</td>
<td>1.5</td>
</tr>
</tbody>
</table>

options are priced with a normal model. With skewed-t marginals and a Power-t copula, however, we price both CMS and CMS spread options consistently with fat-tailed SABR-like distributions.

With the Power-t copula, we also priced the set of spread digitals from the previous sections. In Figure 10, we compare spread digitals calculated with the Power-t copula and skewed-t marginals (via caplet spreads) and the Power-Gaussian copula with normal marginals. We see that in both cases, digital prices are smooth, but slightly different. This is due to the different correlation skews used, cf. Figures 8 and 12. As stated above already, CMS caplets (and therefore also the SABR swaption smile, cf. Section 3.2) are perfectly recovered.

We again shifted the copula parameters by 1% up and down and found similar sensitivities as for the Power-Gaussian copula. For each maturity, a different parameter sensitivity is shown in
Finally, we present in Figure 13 contour plots of the joint distributions that are generated by the calibrated Power-Gaussian copula with normal marginals and the Power-t copula with skewed-t marginals. Darker colours mean higher density values, for really low density values, we plotted black contour lines to better illustrate the tail behaviour.

In case of the Power-Gaussian copula with normal marginals, we see that the probability mass is centered about the diagonal and around the convexity-adjusted CMS rates which is due to the symmetry of the normal marginals. Parameters $\theta_1$ and $\theta_2$ move a small amount of probability mass into the tails (“bulb” effect), and hence induce slightly heavier tails in a specific direction. Since $\theta_1$ and $\theta_2$ can be set independently from each other, they also generate a slight skew.

In case of the Power-t copula with skewed-t marginals, the probability mass is neither centered about the diagonal nor around the convexity-adjusted CMS rates anymore which is due to the skewed-t marginals. With a positive/negative skew, the expected value of a skewed-t density is no longer centered at the peak, cf. figure 1. Therefore, the dark areas in Figure 13 are slightly offset from the diagonal and the convexity-adjusted CMS rates. We further see that due to different skews in the marginals, different shapes of joint densities are generated. We also see that the tails are much heavier than in the normal case and become increasingly heavier with increasing maturity. This is in line with our calibration results, cf. Figure 11, where the degree-of freedom parameter decreases with increasing maturity indicating heavier tails for larger maturities. Parameters $\theta_1$ and $\theta_2$ can no longer shift enough probability mass, thus high strike spread caplet prices are higher than in the normal case.

7 Conclusion

We followed a copula approach to CMS spread option pricing in order to replicate the current standard model, and, as the main result of this paper, to construct a manageable and well-behaved CMS rate joint distribution with fat tails and skew close to a bivariate SABR distribution and consistent with both CMS and CMS spread option markets.

To replicate the standard model, we used a Gaussian copula with normal marginals. We then extended this model in terms of a Power-Gaussian copula, cf. Andersen and Piterbarg 2010. Finally, we introduced our new model, based on a Power-t copula and SABR-like skewed-t marginals, which both have not been used in the spread option literature before. The skewed-t distribution from Fernandez and Steel 1996 is tractable and well-behaved, but can take into account skew and fat tails of the SABR distribution. Power-t copulas are a logical extension of Power-Gaussian
copulas and rich enough to produce a natural joint distribution with skew and heavy tails, but simple enough to maintain tractability.

Extensive numerical tests showed that the standard model recovers spread option prices strike-wise very well, but is inconsistent with the CMS cap market and produces distortions in CMS spread digitals.

We further numerically analysed the extended model, i.e. the effect of a Power-Gaussian copula. The Power-Gaussian copula smoothed out jumps/kinks present in the correlation skew implied by the standard model. As a result, CMS spread digital prices were smooth in the extended model, but the inconsistency to the CMS cap market remained (due to the normal marginals).

Finally, we tested our new model. We found that CMS caps, and as a consequence CMS floors, swaps and SABR swaptions, were accurately recovered (due to the SABR-like skewed-t distribution). For lower maturities, the CMS spread option market could be recovered as well, for higher maturities, high strike spread cap prices were too high, indicating a possible inconsistency in the CMS and CMS spread market.
Figure 11: Fits of skewed-t distributions to 10y and 2y CMS digitals.
Figure 12: Power-t copula calibrated to 10y2y normal correlations and parameter sensitivities.
Figure 13: Joint densities generated by calibrated Power-Gaussian copulas with normal marginals and Power-t copulas with skewed-t marginals.
A Appendix

A.1 Copula Derivatives and Properties

In \((0, 1)^2\), it holds:

\[
\frac{\partial}{\partial u_1} C_G(u_1, u_2) = \Phi \left( \frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u_1)}{\sqrt{1 - \rho^2}} \right)
\]

\[
\frac{\partial^2}{\partial u_1 \partial u_2} C_G(u_1, u_2) = \varphi \left( \frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u_1)}{\sqrt{1 - \rho^2}} \right) \frac{1}{\sqrt{1 - \rho^2} \varphi(\Phi^{-1}(u_2))}
\]

\[
\frac{\partial}{\partial u_1} C_T(u_1, u_2) = F_{d+1} \left( \frac{F_d^{-1}(u_2) - \rho F_d^{-1}(u_1)}{\sqrt{1 - \rho^2}} \sqrt{\frac{d + 1}{d + F_d^{-1}(u_1)^2}} \right)
\]

\[
\frac{\partial^2}{\partial u_1 \partial u_2} C_T(u_1, u_2) = f_{d+1} \left( \frac{F_d^{-1}(u_2) - \rho F_d^{-1}(u_1)}{\sqrt{1 - \rho^2}} \sqrt{\frac{d + 1}{d + F_d^{-1}(u_1)^2}} \right) \frac{1}{\sqrt{1 - \rho^2} d \varphi(F_d^{-1}(u_2))}
\]

\[
\frac{\partial}{\partial u_1} C_P(u_1, u_2) = u_2^{-\theta_2} \left( 1 - \theta_1 ight) u_1^{-\theta_1} C(u_1^{\theta_1}, u_2^{\theta_2}) + \theta_1 \frac{\partial}{\partial u_1} C(u_1^{\theta_1}, u_2^{\theta_2})
\]

\[
\frac{\partial^2}{\partial u_1 \partial u_2} C_P(u_1, u_2) = (1 - \theta_1) u_1^{-\theta_1}(1 - \theta_2) u_2^{-\theta_2} C(u_1^{\theta_1}, u_2^{\theta_2}) + \theta_1 \theta_2 \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1^{\theta_1}, u_2^{\theta_2}) + \theta_2 (1 - \theta_1) u_1^{-\theta_1} \frac{\partial}{\partial u_1} C(u_1^{\theta_1}, u_2^{\theta_2}).
\]

\(\varphi\) denotes the standard normal density, \(f_d\) the Student-t density with \(d\) degrees of freedom. We have almost everywhere:

\(C(u_1, u_2) \in [0, 1]\), nondecreasing

\(C(u, 0) = C(0, u) = 0\)

\(C(u, 1) = C(1, u) = u\)

\(\frac{\partial}{\partial u_1} C(u_1, u_2) \in [0, 1]\), nondecreasing in \(u_2\)

\(\frac{\partial}{\partial u_1} C(u_1, 0) = 0\)

\(\frac{\partial}{\partial u_1} C(u_1, 1) = 1\).

A.2 Proof of Lemma 5.2

Under the \(T_p\)-forward measure, \(S(T)\) has the density \(f_{\mu, \sigma, \epsilon, d}\). We therefore have

\[
\mathbb{E}^{T_p}[\max(S(T) - K, 0)] = \int_{-\infty}^{\infty} \max(x - K, 0) f_{\mu, \sigma, \epsilon, d}(x) dx
\]

\[
= \int_{K}^{\infty} (x - K) f_{\mu, \sigma, \epsilon, d}(x) dx.
\]

We are now left with two cases. When \(K < \mu\):

\[
\int_{K}^{\infty} (x - K) f_{\mu, \sigma, \epsilon, d}(x) dx =
\]

\[
= \frac{2}{\sigma(\epsilon + 1)} \left( \int_{K}^{\mu} (x - K) f_d \left( \frac{\epsilon(x - \mu)}{\sigma \epsilon} \right) dx + \int_{\mu}^{\infty} (x - K) f_d \left( \frac{x - \mu}{\sigma \epsilon} \right) dx \right)
\]

\(24\)
This term vanishes at \( y = \pm \infty \). When \( K \geq \mu \), we similarly get:

\[
\int_{K}^{\infty} (x - K) f_{\mu, \sigma, \epsilon, d}(x) dx =
\]

\[
= \frac{2}{\epsilon + \frac{1}{\epsilon}} \left( \int_{K}^{\infty} (x - K) f_{\mu}(\frac{x - \mu}{\sigma \epsilon}) dx \right)
\]

\[
= \frac{2}{\epsilon + \frac{1}{\epsilon}} \left( \sigma \epsilon^2 \int_{K}^{\infty} x f_{\mu}(x) dx - \varepsilon(K - \mu) \int_{K}^{\infty} f_{\mu}(x) dx \right)
\]

\[
= \frac{2}{\epsilon + \frac{1}{\epsilon}} \left( \sigma \epsilon^2 d + \frac{(K - \mu)^2}{\sigma^2 \epsilon} f_{\mu} \left( \frac{K - \mu}{\sigma \epsilon} \right) - \varepsilon(K - \mu) \left( 1 - F_{\mu} \left( \frac{K - \mu}{\sigma \epsilon} \right) \right) \right).
\]

The convexity-adjusted CMS rate \( S(0) \) can be calculated as a CMS caplet with strike 0.

### A.3 Proof of Lemma 5.4

We follow [Andersen and Piterbarg 2010]. It holds:

\[
V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max(s_1 - s_2 - K, 0) \psi(s_1, s_2) ds_1 ds_2
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s_1 - s_2 - K) I_{s_1 - s_2 - K \geq 0} \psi(s_1, s_2) ds_1 ds_2
\]

\[
= V_1 - V_2.
\]
Then

\[ V_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (s_1 - K) \mathbb{1}_{s_1 - s_2 - K \geq 0} \psi(s_1, s_2) ds_1 ds_2 \]

\[ = \int_{-\infty}^{\infty} (s_1 - K) \psi_1(s_1) \int_{-\infty}^{\infty} \mathbb{1}_{s_1 - s_2 - K \geq 0} \psi_2(s_1|s_2) ds_2 ds_1 \]

\[ = \int_{-\infty}^{\infty} (s_1 - K) \psi_1(s_1) \partial_{u_1} C(\Psi_1(s_1), \Psi_2(s_1 - K)) ds_1 \]

\[ V_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_2 \mathbb{1}_{s_1 - s_2 - K \geq 0} \psi(s_1, s_2) ds_1 ds_2 \]

\[ = \int_{-\infty}^{\infty} s_2 \psi_2(s_2) \int_{-\infty}^{\infty} \mathbb{1}_{s_1 - s_2 - K \geq 0} \psi_1(s_1|s_2) ds_1 ds_2 \]

\[ = \int_{-\infty}^{\infty} s_2 \psi_2(s_2) (1 - Q(S_1(T) \leq s_2 + K|S_2(T) = s_2)) ds_2 \]

\[ = \int_{-\infty}^{\infty} s_2 \psi_2(s_2) \left(1 - \partial_{u_2} C(\Psi_1(s_2 + K), \Psi_2(s_2))\right) ds_2. \]

Above, we used the general copula property

\[ Q(U_2 \leq u_2|U_1 = u_1) = \partial_{u_1} C(u_1, u_2) \]

References


26


V. V. Piterbarg. Spread options, Farkas’s lemma and linear programming. Risk, 2011.