From Minority Game to Black&Scholes pricing

Matteo Ortisi
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From Minority Game to Black&Scholes pricing

Matteo Ortisi*  

Abstract  

In this paper we study the continuum time dynamics of a stock in a market where agents behavior is modeled by a Minority Game with number of strategies for each agent $S = 2$ and “fake” market histories. The dynamics derived is a generalized geometric Brownian motion and an analogous of the Black&Scholes formula for European options is obtained.

Keywords:
Minority Game, Agent-based models, Option pricing, Market calibration.

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1. Introduction

In recent years researchers endeavored to build models that reproduce some empirical statistical regularities of the real financial markets, such as volatility clusters, fat tails, scaling, occurrence of crashes, etc., known as “stylized facts” (see [10, 12, 14]). One possible way to address this problem is the “black-box” stochastic approach. A complex stochastic process which possesses the relevant characteristics of the desired empirical facts is constructed. Process used in this kind of models are the nonlinear diffusion models, the Lévy processes, the jump processes and stochastic volatility models. Another possibility is to follow the agent-based framework, which models the market and derives from the interaction between players the stock price dynamics. Herd Behavior and Minority Game are examples of these models. Choosing between these two models families relies on the purpose of the modeling itself. “black-box” stochastic modeling is used in financial mathematics, where the main goal is the practical use, i.e. pricing of derivatives or portfolio allocation. Agent-based models are mainly used in economics where the theoretical aspects, the explanation and understanding of financial markets, are the main intent. There is a complementarity between these two approaches; indeed “black-box” stochastic models lack of explanatory power by construction and agent-based models can be hardly used for pricing purposes given their complexity. Moreover one of most attractive features of the usual Black&Scholes type models is the possibility of obtaining closed, exact formulas for the pricing of financial derivative securities. Such qualities are fundamental from the point of view of finance practitioners, like financial institutions, that require fast calculations, closed pricing formula for calibration purposes and tools to hedge risky assets.

A very promising way to combine these main issues, is to focus on highly simplified toy models of financial markets relying on the Minority Game (MG) [4]. Variants of this model have been shown to reproduce quite accurately such stylized facts of financial markets (see [1, 3, 5, 11, 19]) and moreover the continuum time limit of the MG provides a bridge between the adherence to empirically observed features of real markets and the practical usability of “black-box” stochastic models, since its evolution follows a system of stochastic differential equations (see [15], [8]).

In this framework, the stock price process is driven by the excess demand or overall market bid. Since the excess demand evolution process has been explicitly obtained in [8] it seems natural to follow this path to obtain a closed pricing formula for financial derivatives. However, as we show in Appendix, the process derived has not independent increments. The fact that the process has not independent increments, not only cause analytical and numerical issue but also conflicts with the Black&Scholes model which is fundamental for financial practitioners. Therefore we decided to use the simplest version of the minority game with “fake” market histories and to recover the excess demand directly from the scores difference stochastic differential equations. Using these approach it is possible to apply the usual “change of measure technique”, that guarantee the absence of arbitrage in the model and the existence of the hedging portfolio, and hence to obtain closed pricing formulas allowing the
model calibration to the market. Undoubtedly the MG with real market history must be investigated and the approach used in [8] must be kept in mind for further development. The work is organized as follow. In the next section the classical MG model and its continuum time limit version are described. In Section 3 we derive the stock price dynamics and in Section 4, by using the classical risk-neutral pricing techniques (see [18]), we apply the Black&Scholes formula to price European options.
2. The Model

In this section the basic concept and the main results of MG useful for our purposes will be exposed (for a comprehensive introduction to MG refer to [6, 9]). Consider the MG with N agents. Its dynamics is defined in terms of dynamical variables $U_{s,i}(t)$; these are scores corresponding to each of the possible agents strategy choices $s = +1, -1$. Each agent takes a decision (strategy choice) $s_i(t)$ with

$$P_{\text{Prob}}\{s_i(t) = s\} = \frac{e^{\Gamma_i U_{s,i}(t)}}{\sum_{s' \in \{-1, 1\}} e^{\sum_{i} \Gamma_i U_{s',i}(t)}}$$

where $\Gamma_i > 0$, and $s' \in \{-1, 1\}$. The original MG corresponds to $\Gamma_i = \infty$ and was generalized to $\Gamma_i = \Gamma < \infty$ [2]; here we consider the latter case.

The public information variable $\mu(t)$, that represent the common knowledge of the past record, is given to all agents; it belongs to the set of integers $\{1, \ldots, P\}$ and can either be the binary encoding of last M winning choices or drawn at random from a uniform distribution:

$$P_{\text{Prob}}\{\mu(t) = \mu\} = \frac{1}{P}, \quad \mu = 1, \ldots, P.$$ 

Here we consider the latter case (see [2]).

The strategies $a_{s,i}^{\mu}$ are uniform random variables taking values $\pm 1$ ($\text{Prob}\{a_{s,i}^{\mu} = \pm 1\} = 1/2$) independent on $i$, $s$ and $\mu$. Here we consider $S = 2$, i.e. 2 strategy for each agent that are randomly drawn at the beginning of the game and kept fixed.

On the basis of the outcome $B(t) = \sum_{i=1}^{N} a_{s,i}^{\mu}(t)$ each agent update his scores according to

$$U_{s,i}(t + \delta t) = U_{s,i}(t) - a_{s,i}^{\mu}(t) \frac{B(t)}{P}, \quad (1)$$

where $\delta t \to 0$ is the time increment.

Let us introduce the following random variables (to ease the notation the choices +1 and -1 are shorted with + and -)

$$\xi_{i}^{\mu} = \frac{a_{+i,i}^{\mu} - a_{-i,i}^{\mu}}{2}, \quad \Theta^{\mu} = \sum_{i=1}^{N} \frac{a_{+i,i}^{\mu} + a_{-i,i}^{\mu}}{2},$$

and their averages

$$\overline{\xi_{i}} \Theta = \frac{1}{P} \sum_{\mu=1}^{P} \xi_{i}^{\mu} \Theta^{\mu}, \quad \overline{\xi_{i}} \xi_{j} = \frac{1}{P} \sum_{\mu=1}^{P} \xi_{i}^{\mu} \xi_{j}^{\mu}.$$
The only relevant quantity in the dynamics is the difference between the scores of the two strategies:

\[ y_\tau(t) = \Gamma \frac{U_{+,\tau}(\tau) - U_{-,\tau}(\tau)}{2}, \]  

where \( \tau = \frac{t}{T} \).

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be the probability space respect to which all our random variables are defined, \( y = (y_i)_{1 \leq i \leq N}, \Theta = (\Theta_i)_{1 \leq i \leq P} \) and \( \xi = (\xi_i)_{1 \leq i \leq N} \).

As shown in [15], if \( P/N = \alpha \in \mathbb{R}_1 \), \( S = 2 \) and \( \Gamma_i = \Gamma > 0 \) for all \( i \), the dynamics of the continuum time limit of the MG is given by the following \( N \)-dimensional stochastic differential equation

\[ dy_i(t) = \left( -\xi_i \Theta - \sum_{j=1}^{N} \xi_i \xi_j \tanh(y_j(t)) \right) dt + A_i(y, \alpha, N, \Gamma, \xi) dW(t), \quad i = 1, \ldots, N \]

where \( W(t) \) is an \( N \)-dimensional Wiener process,

\( A_i \) is the \( i \)-th row of the \( N \times N \) matrix \( A = (A_{ij}) \) such that

\[ (AA')_{ij}(y, \alpha, N, \Gamma, \xi) = \frac{\Gamma \sigma_N^2(y)}{\alpha N} \xi_i \xi_j. \]

The dynamics of \( \sigma_N^2 \) is well known; the relevant feature for our purposes is its limit behavior, as \( N \) grows to infinity, with respect to \( \alpha \): for \( \alpha \geq \alpha_c \), \( \lim_{N \to \infty} \frac{\sigma_N^2(y)}{N} \leq 1, \ \forall y \), while for \( \alpha < \alpha_c \), \( \lim_{N \to \infty} \frac{\sigma_N^2(y)}{N} \leq k, \ \forall y \) and with \( k \) constant. For more details see [9].
3. Stock price dynamics

In this section, using the scores difference stochastic differential equations, the overall market bid and hence the stock price dynamics is derived. Let us consider a single stock in a market modeled by the continuum time MG; following [5], we define the stock price dynamics in terms of excess demand, as

\[
\log p(t + \delta t) = \log p(t) + \frac{B(t)}{N}. \tag{4}
\]

By (1),

\[
\frac{U_{+,i}(t + \delta t) - U_{-,i}(t + \delta t)}{2} = \frac{U_{+,i}(t) - U_{-,i}(t)}{2} - \frac{\mu_{+,i}^{(0)} - \mu_{-,i}^{(0)}}{2} \frac{B(t)}{P}.
\]

In the continuum time limit \( \delta t \to 0 \), we obtain that the scores differences defined by (2) satisfy

\[
dy_i(t) = -\Gamma \xi_i^{(t)} \frac{B(t)}{P}, \quad \forall i = 1, 2, \ldots, N,
\]

or equivalently

\[
dy_i(\Gamma t) = -\Gamma \xi_i^{(t)} \frac{B(t)}{P}, \quad \forall i = 1, 2, \ldots, N. \tag{5}
\]

Since the \( \xi_i^{(t)} \) are independent from \( B(t) \) (the difference between two actions given the past history \( \mu(t) \) is fixed at the beginning of the game and does not depend on the action chosen by the agent on the basis of the score function \( U_{+,i}(t) \)),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left( \xi_i^{(t)} \right)^2 \frac{B(t)}{N} = \mathbb{E} \left[ \left( \xi_i^{(t)} \right)^2 \right] \lim_{N \to \infty} \frac{B(t)}{N} = \frac{1}{2} d \log p(t).
\]

To obtain the time continuum dynamics of the stock price it is hence sufficient to multiply both sides of (5) by \( \xi_i^{(t)} \) and averaging over all the agents \( i \):

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \xi_i^{(t)} dy_i(\Gamma t) = -\frac{1}{\alpha} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left( \xi_i^{(t)} \right)^2 \frac{B(t)}{N},
\]

we obtain that the continuum time dynamics of the stock price is

\[
d \log p(t) = \frac{2\alpha}{\Gamma} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \xi_i^{(t)} dy_i(\Gamma t). \tag{6}
\]
Integrating over \([0,t]\) both sides of (6) and taking into account equation (3), it follows that

\[
p(t) = p(0) \exp \left\{ \lim_{N \to \infty} \int_0^t \frac{2\alpha}{N} \sum_{i=1}^N \left( \xi_i^{\mu(s)} \left( \xi_i \Theta + \sum_{j=1}^N \xi_i \xi_j \tanh(y_j(\Gamma s)) \right) \right) \, ds \right\}
\]

\[
- \lim_{N \to \infty} \int_0^t \frac{2\alpha}{NT} \sum_{i=1}^N \left( \xi_i^{\mu(s)} A_i \right) dW(\Gamma s) \right\}. \tag{7}
\]

The following proposition on the drift and diffusion term of equation (7) holds:

**Proposition 3.1** Let \(t = O(N)\). The drift term of (7) is at most \(O(N)\); the diffusion term is different from zero \(\forall \alpha\), finite for \(\alpha \geq \alpha_c\) and at most \(O(N)\) for \(\alpha < \alpha_c\).

**Proof**

It is easy to see that, by applying the Law of Large Numbers, \(\forall i \lim_{N \to \infty} \xi_i = 0\); \(\lim_{N \to \infty} \xi_i \xi_j = 0\) and \(\lim_{N \to \infty} \xi_i \Theta = 0\) (for detailed computations see [17]); it follows that

\[
\lim_{N \to \infty} \frac{2\alpha}{N} \sum_{i=1}^N \left( \xi_i^{\mu(s)} \left( \xi_i \Theta + \sum_{j=1}^N \xi_i \xi_j \tanh(y_j(\Gamma s)) \right) \right) = \alpha \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \xi_i^{\mu(s)} \tanh(y_i(\Gamma s)) < k\alpha,
\]

with \(k\) finite constant. Hence the drift term is at most \(k\alpha N\) and, due to different initial conditions and the stochastic evolution of \(y_i\), in general different from zero.

Consider the zero mean random variable

\[
Y = \int_0^t \frac{2\alpha}{NT} \sum_{i=1}^N \left( \xi_i^{\mu(s)} A_i \right) dW(\Gamma s);
\]

since

\[
\frac{2\alpha}{NT} \sum_{i=1}^N \left( \xi_i^{\mu(s)} A_i \right) dW(\Gamma s) = \sum_{j=1}^N \left( \frac{2\alpha}{NT} \sum_{i=1}^N \xi_i^{\mu(s)} A_{ij} \right) dW_j(\Gamma s),
\]

\(Y\) has variance

\[
\nu = \int_0^t \sum_{j=1}^N \left( \frac{2\alpha}{NT} \sum_{i=1}^N \xi_i^{\mu(s)} A_{ij} \right)^2 d(\Gamma s)
\]

\[
= \int_0^t \frac{4\alpha^2}{N^2} \sum_{i=1}^N \left( \xi_i^{\mu(s)} \right)^2 (A A')_{ii} ds + \int_0^t \frac{4\alpha^2}{N^2} \sum_{i,j=1,i\neq j}^N \xi_i^{\mu(s)} \xi_j^{\mu(s)} (A A')_{ij} ds. \tag{9}
\]

As \(N\) grows to infinity, by applying the Law of Large Numbers, \(\lim_{N \to \infty} A A' = \lim_{N \to \infty} \frac{1}{\sqrt{N^2}} I\), where \(I\) is the identity \(N \times N\) matrix (see [17] for details), and the first term of the right hand side of (9) is equal to \(\lim_{N \to \infty} \frac{\alpha^2}{N^2} < 0\) while the second one is equal to 0. Since for \(\alpha \geq \alpha_c\), \(0 < \frac{\alpha^2}{N^2} < 1\) and for \(\alpha < \alpha_c\), \(0 < \frac{\alpha^2}{N^2} < kN\), the thesis follows.
Consider the dynamics

\[ q(t) = p(0) \exp \left\{ \int_0^t \frac{2\alpha}{N} \sum_{i=1}^N \left[ \epsilon_i^{(t)} \left( \xi_i \Theta + \sum_{j=1}^N \xi_i \xi_j \tanh(y_j(\Gamma s)) \right) \right] ds \right. \]

\[ - \int_0^t \frac{2\alpha}{NT} \sum_{i=1}^N \left( \epsilon_i^{(t)} A_i \right) dW(\Gamma s) \left\} ; \]

by definition \( q(t) \) converges pointwise towards \( p(t) \) (\( \lim_{N \to \infty} q(t) = p(t) \), \( \forall \omega \in \Omega \)) and hence almost surely; it follows that a.s. \( \forall \epsilon > 0 \) there exists \( N_0 \) such that, \( \forall N > N_0 \), \( |p(t) - q(t)| < \epsilon \). By assuming \( t = kN \) and \( N \) finite but sufficiently large, we have that a.s. \( q(t) \sim p(t) \) and proposition 3.1 holds for \( q(t) \).

From now on we assume \( t = kN \) and \( N \) finite but sufficiently large to have

\[ p(t) = p(0) \exp \left\{ \int_0^t \frac{2\alpha}{N} \sum_{i=1}^N \left[ \epsilon_i^{(t)} \left( \xi_i \Theta + \sum_{j=1}^N \xi_i \xi_j \tanh(y_j(\Gamma s)) \right) \right] ds \right. \]

\[ - \int_0^t \frac{2\alpha}{NT} \sum_{i=1}^N \left( \epsilon_i^{(t)} A_i \right) dW(\Gamma s) \left\} ; \]

\[ (10) \]

it is worth to note that as a consequence of proposition 3.1, equation (10) has a unique solution.

Let us define \( \mathcal{G}_t = \sigma(W(\Gamma s), \xi^{(t)}(s); s \leq t) \) the filtration generated by both the processes \( W \) and \( \xi \). The drift and the diffusion terms of equation (10) are adapted to the filtration \( \mathcal{G}_t \) (\( \xi_i \Theta \) and \( \xi_i \xi_j \) are non random while \( y_j(\Gamma s) \) because of its dynamics (2)).

The differential of the stock price process is therefore, by Ito’s formula,

\[ \frac{dp(t)}{p(t)} = \left[ \frac{2\alpha}{N} \sum_{i=1}^N \xi_i^{(t)} \left( \xi_i \Theta + \sum_{j=1}^N \xi_i \xi_j \tanh(y_j(\Gamma t)) \right) \right] dt
\]

\[ + \frac{2\alpha^2}{\Gamma N^2} \left( \sum_{i=1}^N \xi_i^{(t)} A_i \right) \left( \sum_{i=1}^N \xi_i^{(t)} A_i \right)^T dt
\]

\[ - \frac{2\alpha}{NT} \sum_{i=1}^N \left( \xi_i^{(t)} A_i \right) dW(\Gamma t) \]

\[ = \left[ \frac{2\alpha}{N} \sum_{i=1}^N \xi_i^{(t)} \left( \xi_i \Theta + \sum_{j=1}^N \xi_i \xi_j \tanh(y_j(\Gamma t)) \right) + \frac{2\alpha^2}{\Gamma N^2} \sum_{i=1}^N \left( \sum_{i=1}^N \xi_i^{(t)} A_{ii} \right)^2 \right] dt
\]

\[ - \frac{2\alpha}{NT} \sum_{i=1}^N \left( \xi_i^{(t)} A_i \right) dW(\Gamma t) . \]
4. Derivative security pricing

In this section we develop the risk-neutral pricing of a derivative security on a single stock whose price dynamics is given by (10). We follow the usual scheme: construction of the risk-neutral measure, relying on Girsanov Theorem, for the discounted process and, by using the Martingale Representation Theorem, of the replication portfolio that allows to hedge a short position in the derivative security (see [18]). At the end of the section we apply Black&Scholes formula to price a European call option.

4.1 Discounted stock dynamics under risk-neutral measure

Consider the interest rate process $R(\omega)$ adapted to the filtration $\mathcal{F}_t$, $0 \leq t \leq T$. The discount process $D(t) = e^{-\int_0^t R(\omega) d\omega}$ has, by Ito’s formula, differential $dD(t) = -R(t) D(t) dt$. The discounted stock process is

$$D(t)p(t) = p(0) \exp \left\{ \int_0^t -R(s) + \frac{2\alpha}{N} \sum_{i=1}^N \xi_i^{\mu} \left( \xi_i \Theta + \sum_{j=1}^N \xi_i \xi_j \tanh(y_j(\Gamma s)) \right) ds \right\}$$

and its differential

$$\frac{d(D(t)p(t))}{D(t)p(t)} = \left[ \frac{2\alpha}{N} \sum_{i=1}^N \xi_i^{\mu} \left( \xi_i \Theta + \sum_{j=1}^N \xi_i \xi_j \tanh(y_j(\Gamma s)) \right) \right]$$

$$+ \frac{2\alpha^2}{\Gamma N^2} \sum_{j=1}^N \left( \sum_{i=1}^N \xi_i^{\mu} A_{ij} \right)^2 - R(t) dt - \frac{2\alpha}{NT} \sum_{i=1}^N \left( \xi_i^{\mu} A_{i} \right) dW(\Gamma t).$$

**Proposition 4.1** If $T = O(N)$ there exists a probability measure $\widetilde{\mathbb{P}}$ (risk neutral measure) under which the discounted stock price $D(t)p(t)$ is a $\widetilde{\mathbb{P}}$-martingale.
Proof

Let us define the market price of risk equation to be

$$\frac{2\alpha}{NT} \sum_{i=1}^{N} \left( \xi_{i}^{(u)} A_{i} \right) \gamma(\Gamma t) = \left[ \frac{2\alpha}{N} \sum_{i=1}^{N} \left( \xi_{i}^{(s)} \left( \xi_{i} \Theta + \sum_{j=1}^{N} \xi_{i} \xi_{j} \tanh(y_{j}((\Gamma s))) \right) \right) \right] +$$

$$+ \frac{2\alpha^{2}}{\Gamma N^{2}} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} \left( \xi_{i}^{(t)} A_{ij} \right) \right)^{2} - R(t) \right],$$

where $\gamma(\Gamma t) = (\gamma_{1}(\Gamma t), \ldots, \gamma_{N}(\Gamma t))$ is an unknown process; the differential of the discounted process becomes

$$d(D(t)p(t)) = D(t)p(t) \frac{2\alpha}{NT} \sum_{i=1}^{N} \left( \xi_{i}^{(u)} A_{i} \right) [\gamma(\Gamma t) dt - dW(\Gamma t)].$$

Let $|| \cdot ||$ denote the usual Euclidean norm; we show that for $T = O(N), \int_{0}^{T} ||\gamma(\Gamma u)||^2 d(\Gamma u) < \infty$. Consider $N$ finite and sufficiently large; since

$$\int_{0}^{T} \left| \left| \frac{2\alpha}{NT} \sum_{i=1}^{N} \left( \xi_{i}^{(u)} A_{i} \right) \right| \right|^2 d(\Gamma u) = \int_{0}^{T} \sum_{j=1}^{N} \left( \frac{2\alpha}{N} \sum_{i=1}^{N} \left( \xi_{i}^{(u)} \right) \right)^2 d(\Gamma u) > 0 \text{ and finite},$$

$$\int_{0}^{T} \left| \left| \frac{2\alpha}{N} \sum_{i=1}^{N} \left( \xi_{i}^{(s)} \left( \xi_{i} \Theta + \sum_{j=1}^{N} \xi_{i} \xi_{j} \tanh(y_{j}((\Gamma s))) \right) \right) \right| \right|^2 d(\Gamma u) < \infty,$$

and

$$\int_{0}^{T} \left| \left| \frac{2\alpha^{2}}{\Gamma N^{2}} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} \left( \xi_{i}^{(t)} A_{ij} \right) \right) \right| \right|^2 d(\Gamma u) \sim 0,$$

$$0 < \int_{0}^{T} ||\gamma(\Gamma u)||^2 d(\Gamma u) < \infty.$$

It follows that $\gamma(\Gamma t)$ satisfies the Novikov condition (see [16])

$$E \left[ \frac{1}{2} \int_{0}^{T} ||\gamma(\Gamma u)||^2 du \right] < \infty.$$

As a consequence we can apply Girsanov’s Theorem (see [13]); there exists a probability measure $\tilde{P}$ (risk neutral measure) under which $\tilde{W}(\Gamma t) = \int_{0}^{T} \gamma(\Gamma u) du - W(\Gamma t)$ is an $N$-dimensional Brownian motion. It follows that

$$d(D(t)p(t)) = D(t)p(t) \frac{2\alpha}{NT} \sum_{i=1}^{N} \left( \xi_{i}^{(t)} A_{i} \right) d\tilde{W}(\Gamma t),$$

and hence the discounted stock price is a $\tilde{P}$-martingale.
By making the replacement \( dW(\Gamma_t) = \gamma(\Gamma_t) dt - d\tilde{W}(\Gamma_t) \), dynamics \( (10) \) of the undiscounted stock price \( p(t) \) becomes

\[
\begin{align*}
p(t) &= p(0) \exp \left\{ \int_0^t R(s) - \frac{2\alpha^2}{N^2} \sum_{j=1}^N \left( \sum_{i=1}^N \xi_i(t) A_{ij} \right)^2 ds + \right. \\
&\left. \int_0^t \frac{2\alpha}{NT} \sum_{i=1}^N \left( \xi_i^{(t)} A_i \right) d\tilde{W}(\Gamma_s) \right\}. & (12)
\end{align*}
\]

### 4.2 European call option pricing

Since, for \( T - t = O(N) \), there exists a risk neutral measure \( \mathbb{P} \) for the discounted process \( D(t)p(t) \), we can apply the usual scheme, relying on the martingale representation theorem, to show the existence of a replication portfolio for a derivative security pricing.

Consider \( T - t = O(N) \); let \( V(T) \) be an \( \mathcal{F}_T \)-measurable random variable representing the payoff at time \( T \) of a derivative security. The process \( E(t) = \mathbb{E}_\mathbb{P}[D(T)V(T)|\mathcal{F}_t] \) is a \( \mathbb{P} \)-martingale (it follows from iterated conditioning); by the Martingale Representation Theorem there exists an initial capital \( X(0) \) and a portfolio strategy \( \Delta(t) \) such that \( X(T) = V(T) \) almost surely and an adapted process \( \phi(t) \) which constructs \( E(t) \) out of \( D(t)p(t) \). It follows that \( D(t)X(t) \) is a \( \mathbb{P} \)-martingale

\[
D(t)X(t) = \mathbb{E}_\mathbb{P}[D(T)X(T)|\mathcal{F}_t] = \mathbb{E}_\mathbb{P}[D(T)V(T)|\mathcal{F}_t]. & (13)
\]

The value \( X(t) \) of the hedging portfolio in \( (13) \) is the capital needed at time \( t \) in order to successfully hedge the position in the derivative security with payoff \( V(T) \). Hence \( V(t) \) is the price of the derivative security at time \( t \) and we obtain the usual risk neutral pricing formula (see [18])

\[
D(t)V(t) = \mathbb{E}_\mathbb{P}[D(T)V(T)|\mathcal{F}_t], \quad 0 \leq t \leq T;
\]

by recalling the definition of \( D(t) \),

\[
V(t) = \mathbb{E}_\mathbb{P}[e^{-\int_0^T R(s) ds} V(T)|\mathcal{F}_t], \quad 0 \leq t \leq T.
\]

**Proposition 4.2** Let us assume constant interest rate \( r \); the price of a European call option with underlying \( p(t) \) and strike \( K \) is

\[
\begin{align*}
\text{c}(t, p(t); K, r, \nu) &= p(t) \psi(\delta(t, p(t))) - e^{-r\theta} K \psi (\delta(t, p(t)) - \sqrt{\nu}), \\
\end{align*}
\]

where \( \theta = T - t, \nu = \frac{\alpha \beta}{N^2}, \delta(t, p(t)) = \frac{[\log \frac{p(t)}{K} + (r + \nu \beta) \delta]}{\sqrt{\nu}} \) and \( \psi \) the erf function.
Proof

Consider the random variable

\[ Y = \int_t^T 2\alpha \frac{1}{NT} \sum_{i=1}^N \left( \xi^{\mu(s)}_i A_i \right) d\tilde{W}(T s) = \sum_{j=1}^N \int_t^T 2\alpha \frac{1}{NT} \sum_{i=1}^N \left( \xi^{\mu(s)}_i A_{ij} \right) dW_j(T s). \]

\( Y \) has zero mean and, in the limit as \( N \) grows to infinity, finite variance \( \nu \) (see prop. 3.1); since we consider \( N \) sufficiently large, we can assume \( \text{Var}[Y] = \nu \). By applying the Law of Large Numbers we can assume, for \( N \) sufficiently large, \( \frac{1}{N} \sum_{i=1}^N \left( \xi^{\mu(s)}_i A_{ij} \right) \) to be not random; it follows that \( Y \) has a normal distribution with 0 mean and variance \( \nu = \frac{\alpha \sigma_j^2 (T-t)}{N^2} \) and equation (12) becomes

\[ p(T) = p(t) \exp \left\{ \left( r - \frac{1}{2} \right) \theta - \frac{\nu}{\theta} \sqrt{\theta} Z \right\}, \]

\( \theta = T - t \) and \( Z = -\frac{\nu}{\sqrt{\theta}} \) is a standard normal random variable.

We can apply Black-Scholes formula (see [18]) and obtain the price of a call option with underlying \( p \) and strike \( K \):

\[ c(t, p(t); K, r, \nu) = p(t) \psi(d(\theta, p(t))) - e^{-\theta} K \psi \left( d(\theta, p(t)) - \sqrt{\nu} \right), \]

where \( d(\theta, p(t)) = \frac{\left[ \log \left( \frac{p(t)}{K} \right) + (r + \frac{1}{2} \nu) \theta \right]}{\sqrt{\nu}} \) and \( \psi \) the erf function.
5. Conclusion

In this paper we have shown that under the assumption of the MG with "fake" market histories as traders interactions model, the stock price dynamics follows the generalized exponential Brownian motion. This allowed us to apply the usual Black&Scholes pricing formula to value call/put European options. The next step should be the calibration, in terms of the game parameter of the MG on the real options market; through the calibration would be possible to understand which part of the spectrum is coherent with the real market.
6. Appendix

In this section we recover the stock price dynamics relying directly on the bid evolution process by using its explicit form obtained in [8]. The key point is that such process has dependent increments that makes hard to develop the usual framework to obtain closed option pricing formulas.

As showed in [8] the bid evolution process at time step \( l \) has the following explicit form:

\[
B(l) = \phi(l) - \frac{1}{2} \bar{\eta} \sum_{l' \neq l} G(l, l') \delta_{\lambda(l, B, Z), \lambda(l', B, Z)} B(l'),
\]

(16)

With the zero-mean Gaussian random field \( \{ \phi \} \), characterized by

\[
\langle \phi_l, \phi_{l'} \rangle = \frac{1}{2} [1 + C(l, l')] \delta_{\lambda(l, B, Z), \lambda(l', B, Z)} B(l'),
\]

(17)

where \( C(l, l') \) and \( G(l, l') \) are non-random coefficients defined as averages over disorder variable and \( \lambda(l, B, Z) \) represents historic market data available at time \( l \); in the simplest case of a MG with “fake” market histories \( \lambda(l, B, Z) = \mu(l) \).

The single stock price dynamics satisfies

\[
\log \frac{p(m + 1)}{p(0)} = \frac{1}{N} \sum_{i=0}^{m} A(i).
\]

In order to carry out the temporal coarse-graining and transform the dynamics to an appropriately rescaled continuous time \( t \) we consider the canonical time rescaling where each time step has a duration \( \delta_N = \frac{1}{N} \), which will be real valued in the limit \( N \to \infty \) (see[7]). The present canonical definition of the time unit \( t \) implies temporal coarse-graining over \( O(1/\delta_N) = O(N) \) iteration steps and (4) becomes

\[
\log \frac{p(t)}{p(0)} = Y(t),
\]

where

\[
Y(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} A(kt\delta_N).
\]
The stock price process is driven by the \{Y(t)\}_t process; as a consequence, in order to price an option on a stock, we have to investigate the properties of \{Y(t)\}_t as stochastic processes that from now on we suppose to be defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The following properties deal with such goal.

The first observation is that \(Y(t) < \infty\). It follows from \(\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{1/\delta_N-1} A^2(kt\delta_N) = \sigma^2\) (see [7]).

**Proposition 6.1** \(\{Y(t)\}_t\) is not a martingale.

**Proof**

Let us assume that \(\{Y(t)\}_t\) is a martingale with respect to a filtration \((\mathcal{F}_t)_t\). By equation (16) we have

\[
A(0) = \phi(0)
\]

\[
A(t\delta_N) = \phi(t\delta_N) - \frac{1}{2} \tilde{G}(t\delta_N, 0) \delta_{\lambda(t\delta_N), \lambda(0)} \phi(0),
\]

\[
A(2t\delta_N) = \phi(2t\delta_N) - \frac{1}{2} \tilde{G}(2t\delta_N, \delta_{\lambda(2t\delta_N), \lambda(t\delta_N)} \phi(0),
\]

\[
+ \frac{1}{4} \tilde{G}(2t\delta_N, \delta_{\lambda(2t\delta_N), \lambda(t\delta_N), \lambda(0)} \phi(0),
\]

\[
+ \frac{1}{2} \tilde{G}(2t\delta_N, 0) \delta_{\lambda(2t\delta_N), \lambda(0)} \phi(0),
\]

\[\vdots\]

Since the \(G(kt\delta_N, k't\delta_N)\) coefficients are deterministic and in the MG with “fake” market histories \(\delta_{\lambda(kt\delta_N), \lambda(k't\delta_N)}\) are independent from the \(\phi\) (they depend only on the random variables \(Z_t\)) and hence \(\mathbb{E}[\delta_{\lambda(kt\delta_N), \lambda(k't\delta_N)} \phi(k't\delta_N)] = \mathbb{E}[\delta_{\lambda(kt\delta_N), \lambda(k't\delta_N)}] = 0\), it is easy to see that each \(A(kt\delta_N), k = 0, \ldots, 1/\delta_N - 1\), is a linear combination of zero mean normal random variable \(\phi\); it follows that \(\mathbb{E}[Y(t)] = 0\).

Consider the random variable \(I(t, s) = Y(t) - Y(s)\) with \(0 \leq u \leq s \leq t\) generic time instants.

\[
\text{Cov}(I(t, s), Y(u)) = \mathbb{E}[I(t, s) - \mathbb{E}[I(t, s)])(Y(u) - \mathbb{E}[Y(u)])] = \mathbb{E}[I(t, s)Y(u)]
\]

\[= \mathbb{E}[\mathbb{E}[I(t, s)Y(u)|\mathcal{F}_u]] = \mathbb{E}[Y(u)\mathbb{E}[I(t, s)|\mathcal{F}_u]] = 0,
\]

where we have used the fact that \(Y(u)\) is \(\mathcal{F}_u\)-measurable and finite and that \(\{Y(t)\}_t\) is a martingale.

But since \(\{\phi\}\) is a Gaussian field with covariance (17), there exist \(0 \leq u \leq s \leq t\) such that \(\text{Cov}(I(t, s), Y(u)) \neq 0\) and we can conclude that \(\{Y(t)\}_t\) is not a martingale. \(\square\)

It is worth to note that, being \(Y(t)\) normally distributed, \(\text{Cov}(I(t, s), Y(u)) \neq 0\) implies that \(\{Y(t)\}_t\) has dependent increments (we remember that \(\{Y(t)\}_t\) has independent increments if \(\forall t > s, Y(t) - Y(s)\) is independent from \(\sigma(Y(u), u \leq s)\). While in the usual Black& Scholes framework the stochastic term of the stock price dynamics is a Wiener process and the “non-martingality”, only due to the presence of a drift term, can easily eliminated with a change of measure, here the situation looks like more complicated.
References


